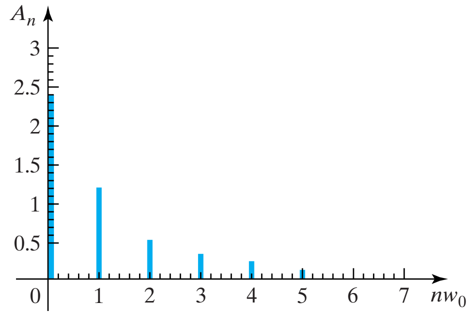


FIGURE 9.13 The amplitude spectrum of $f(x)$ as a function of frequency.

The first few numerical values of the amplitudes are

$$A_0 = 2.356, \quad A_1 = 1.185, \quad A_2 = 0.5, \quad A_3 = 0.341, \quad A_4 = 0.25, \quad A_5 = 0.202,$$
$$A_6 = 0.167, \ldots,$$

and the amplitude spectrum of $f(x)$ is shown in Fig. 9.13. In Fig. 9.13 the amplitudes $A_0, A_1, \ldots,$ are represented by vertical lines of length $A_0, A_1, \ldots,$ corresponding to the frequencies $0, 1, 2, \ldots$.

The phases $\delta_n = \text{Arctan}\,(-b_n/a_n)$ are seen to be given by

$$\delta_1 = \text{Arctan}\,(\pi/2), \quad \delta_2 = \text{Arctan}\,(-\infty), \quad \delta_3 = \text{Arctan}\,(3\pi/2),$$
$$\delta_4 = \text{Arctan}\,(-\infty), \quad \delta_5 = \text{Arctan}\,(5\pi/2), \ldots.$$

The negative sign is required in the arctangent functions associated with phases with even suffixes so that when the terms $A_{2n}\cos(2nx + \delta_{2n})$ are expanded, the functions $\sin 2nx$ have a positive sign. ■

Summary

It was shown how a Fourier series can be interpreted in a different way by introducing an angular frequency ω_0, combining sine and cosine terms with similar arguments into a single cosine term with a phase angle, and calling the magnitude of the multiplier of the cosine term the amplitude associated with the cosine term. A discrete plot of amplitude as a function of frequency was then called the amplitude spectrum of the representation. This form of representation is useful in many applications involving vibrations, because when the response of a system is represented in this way, the square of the amplitude is proportional to the energy in the system at that frequency, so the plot shows the distribution of energy as a function of frequency.

EXERCISES 9.5

In the following exercises find the frequency and amplitude spectrum of the given functions.

1. $f(x) = \begin{cases} 0, & -2\pi < x < 0 \\ x, & 0 < x < 2\pi. \end{cases}$

2. $f(x) = x, \quad -\pi/2 < x < \pi/2.$

3. $f(x) = \begin{cases} 1, & -\pi < x < 0 \\ -3, & 0 < x < \pi. \end{cases}$

4. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi. \end{cases}$

5. $f(x) = x^2, \quad -\pi/4 < x < \pi/4.$

9.6 Double Fourier Series

Fourier series representations extend in a natural way to functions $f(x, y)$ of two real variables x and y over the intervals $-L_1 \leq x \leq L_1$ and $-L_2 \leq y \leq L_2$, provided f can be represented as a Fourier series in x when y is held constant, and as a Fourier series in y when x is held constant.

extending Fourier series to function $f(x, y)$ of two variables

To arrive at a double Fourier series representation for $f(x, y)$, we first consider y to be a constant and write $f(x, y)$ as

$$f(x, y) = \sum_{m=0}^{\infty} \left(A_m(y) \cos \frac{m\pi x}{L_1} + B_m(y) \sin \frac{m\pi x}{L_1} \right), \tag{49}$$

and then allow y to vary by replacing the Fourier coefficients $A_m(y)$ and $B_m(y)$ by their Fourier series representations

$$A_m(y) = \sum_{n=0}^{\infty} \left(a_{mn} \cos \frac{n\pi y}{L_2} + b_{mn} \sin \frac{n\pi y}{L_2} \right) \tag{50}$$

and

$$B_m(y) = \sum_{n=0}^{\infty} \left(c_{mn} \cos \frac{n\pi y}{L_2} + d_{mn} \sin \frac{n\pi y}{L_2} \right).$$

Substituting (50) into (49) shows $f(x, y)$ can be written as

$$\begin{aligned} f(x, y) = &\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(a_{mn} \cos \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2} + b_{mn} \cos \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2} \right) \\ &+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(c_{mn} \sin \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2} + d_{mn} \sin \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2} \right). \end{aligned} \tag{51}$$

The Fourier coefficients a_{mn} for $m, n = 1, 2, \ldots$ are found by multiplying (51) by $\cos \frac{s\pi x}{L_1}$ and integrating over the interval $-L_1 \leq x \leq L_1$ to get

$$\begin{aligned} \int_{-L_1}^{L_1} f(x, y) \cos \frac{s\pi x}{L_1} dx = &\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[a_{mn} \cos \frac{n\pi y}{L_2} \int_{-L_1}^{L_1} \cos \frac{m\pi x}{L_1} \cos \frac{s\pi x}{L_1} dx \right] \\ &+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[b_{mn} \sin \frac{n\pi y}{L_2} \int_{-L_1}^{L_1} \cos \frac{m\pi x}{L_1} \cos \frac{s\pi x}{L_1} dx \right] \\ &+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[c_{mn} \cos \frac{n\pi y}{L_2} \int_{-L_1}^{L_1} \sin \frac{m\pi x}{L_1} \cos \frac{s\pi x}{L_1} dx \right] \\ &+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[d_{mn} \sin \frac{n\pi y}{L_2} \int_{-L_1}^{L_1} \sin \frac{m\pi x}{L_1} \cos \frac{s\pi x}{L_1} dx \right]. \end{aligned} \tag{52}$$

The orthogonality of the functions $\cos \frac{m\pi x}{L_1}$ and $\sin \frac{s\pi x}{L_1}$ over the interval $-L_1 \leq x \leq L_1$ reduces (52) to

$$\int_{-L_1}^{L_1} f(x, y) \cos \frac{s\pi x}{L_1} dx = \sum_{n=0}^{\infty} \left(a_{sn} L_1 \cos \frac{n\pi y}{L_2} + b_{sn} L_1 \sin \frac{n\pi y}{L_2} \right). \tag{53}$$

Multiplication of (53) by $\cos \frac{t\pi y}{L_2}$ followed by integration over the interval $-L_2 \leq$

$y \le L_2$ reduces it further to

$$\int_{-L_2}^{L_2}\left[\int_{-L_1}^{L_1} f(x, y)\cos\frac{s\pi x}{L_1}\right]\cos\frac{t\pi y}{L_2}dy = a_{st}L_1L_2,$$

so replacing s by m and t by n gives

$$a_{mn} = \frac{1}{L_1L_2}\int_{-L_2}^{L_2}\int_{-L_1}^{L_1} f(x, y)\cos\frac{m\pi x}{L_1}\cos\frac{n\pi y}{L_2}dxdy \quad \text{for } m, n = 1, 2, \ldots. \tag{54}$$

The coefficient a_{00} follows by setting $m = n = 0$ in (51) and integrating over the intervals $-L_1 \le x \le L_1$ and $-L_2 \le y \le L_2$ to give

$$a_{00} = \frac{1}{4L_1L_2}\int_{-L_2}^{L_2}\int_{-L_1}^{L_1} f(x, y)dxdy. \tag{55}$$

It remains to find the coefficients a_{m0} and a_{0n} for $m, n = 1, 2, \ldots$. Setting $n = 0$ in (53), integrating over $-L_2 \le y \le L_2$, and then replacing s by m gives

$$a_{m0} = \frac{1}{2L_1L_2}\int_{-L_2}^{L_2}\int_{-L_1}^{L_1} f(x, y)\cos\frac{m\pi x}{L_1}dxdy. \tag{56}$$

The coefficients a_{0n} for $n = 1, 2, \ldots$ follow by multiplying (51) by $\cos\frac{t\pi y}{L_1}$, integrating over the interval $-L_2 \le y \le L_2$, and then replacing t by n to obtain

$$a_{0n} = \frac{1}{2L_1L_2}\int_{-L_2}^{L_2}\int_{-L_1}^{L_1} f(x, y)\cos\frac{n\pi y}{L_2}dxdy. \tag{57}$$

Corresponding arguments show that for $m, n = 1, 2, \ldots,$

$$b_{mn} = \frac{1}{L_1L_2}\int_{-L_2}^{L_2}\int_{-L_1}^{L_1} f(x, y)\cos\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}dxdy, \tag{58}$$

$$c_{mn} = \frac{1}{L_1L_2}\int_{-L_2}^{L_2}\int_{-L_1}^{L_1} f(x, y)\sin\frac{m\pi x}{L_1}\cos\frac{n\pi y}{L_2}dxdy, \tag{59}$$

$$d_{mn} = \frac{1}{L_1L_2}\int_{-L_2}^{L_2}\int_{-L_1}^{L_1} f(x, y)\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}dxdy, \tag{60}$$

where

$$b_{m0} = 0, \quad c_{0n} = 0, \quad d_{0n} = 0 \quad \text{and} \quad d_{m0} = 0, \tag{61}$$

because the index zero causes the sine function to vanish in the integrands of the integrals defining these constants.

general and special double Fourier series representations

Thus, the general **double Fourier series representation** of $f(x, y)$ over the interval $-L_1 \le x \le L_1$ and $-L_2 \le y \le L_2$ is given by

$$\begin{aligned} f(x, y) = &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{mn}\cos\frac{m\pi x}{L_1}\cos\frac{n\pi y}{L_2} + b_{mn}\cos\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}\right) \\ &+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(c_{mn}\sin\frac{m\pi x}{L_1}\cos\frac{n\pi y}{L_2} + d_{mn}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}\right), \end{aligned} \tag{62}$$

where the coefficients a_{mn}, b_{mn}, c_{mn}, and d_{mn} are given by expressions (54) to (61).

The following useful special cases arise according as the function $f(x, y)$ is even or odd in its variables.

Case (a) $f(x, y)$ Is Even in x and y

In this case $f(-x, y) = f(x, y)$ and $f(x, -y) = f(x, y)$, so only the coefficients a_{mn} are nonzero, leading to the **double Fourier cosine series representation**

$$\begin{aligned} f(x, y) = a_{00} &+ \sum_{m=1}^{\infty} a_{m0} \cos \frac{m\pi x}{L_1} + \sum_{n=1}^{\infty} a_{0n} \cos \frac{n\pi y}{L_2} \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \cos \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2}. \end{aligned} \tag{63}$$

As $f(x, y)$ is even in both x and y, both limits of integration in the integrals defining the a_{mn} in (54) to (57) can be changed to give

$$\begin{aligned} a_{00} &= \frac{1}{L_1 L_2} \int_0^{L_2} \int_0^{L_1} f(x, y) dx dy \\ a_{m0} &= \frac{2}{L_1 L_2} \int_0^{L_2} \int_0^{L_1} f(x, y) \cos \frac{m\pi x}{L_1} dx dy, \quad m = 1, 2, \ldots \\ a_{0n} &= \frac{2}{L_1 L_2} \int_0^{L_2} \int_0^{L_1} f(x, y) \cos \frac{n\pi y}{L_2} dx dy, \quad n = 1, 2, \ldots \\ a_{mn} &= \frac{4}{L_1 L_2} \int_0^{L_2} \int_0^{L_1} f(x, y) \cos \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2} dx dy, \quad m, n = 1, 2, \ldots. \end{aligned} \tag{64}$$

Case (b) $f(x, y)$ Is Even in x and Odd in y

In this case $f(-x, y) = f(x, y)$ and $f(x, -y) = -f(x, y)$ so only the coefficients b_{mn} are nonzero, leading to the representation

$$f(x, y) = \sum_{n=1}^{\infty} b_{0n} \sin \frac{n\pi y}{L_2} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \cos \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2}. \tag{65}$$

As $f(x, y)$ is even only in x, the limits of integration for x in integral (58) defining the coefficients b_{mn} can be changed to give

$$\begin{aligned} b_{mn} &= \frac{2}{L_1 L_2} \int_{-L_2}^{L_2} \int_0^{L_1} f(x, y) \cos \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2} dx dy \\ &= \frac{4}{L_1 L_2} \int_0^{L_2} \int_0^{L_1} f(x, y) \cos \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2} dx dy. \end{aligned} \tag{66}$$

Case (c) $f(x, y)$ Is Odd in x and Even in y

In this case $f(-x, y) = -f(x, y)$ and $f(x, -y) = f(x, y)$, so only the coefficients c_{mn} are nonzero, leading to the representation

$$f(x, y) = \sum_{m=1}^{\infty} c_{m0} \sin \frac{m\pi y}{L_1} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn} \sin \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2}. \tag{67}$$

As $f(x, y)$ is even only in y, the limits of integration for y in integral (59) defining the coefficients c_{mn} can be changed to give

$$\begin{aligned} c_{mn} &= \frac{2}{L_1 L_2} \int_{-L_2}^{L_2} \int_{-L_1}^{L_1} f(x, y) \sin \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2} dx dy \\ &= \frac{4}{L_1 L_2} \int_0^{L_2} \int_{-L_1}^{L_2} f(x, y) \sin \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2} dx dy. \end{aligned} \tag{68}$$

Case (d) $f(x, y)$ Is Odd in x and y

In this case $f(-x, y) = -f(x, y)$ and $f(x, -y) = -f(x, y)$ so only the coefficients d_{mn} are nonzero, leading to the **double Fourier sine series representation**

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} d_{mn} \sin \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2}. \tag{69}$$

As $f(x, y)$ is odd in both x and y, both limits of integration for x and y in integral (60) defining the coefficients d_{mn} can be changed to give

$$d_{mn} = \frac{4}{L_1 L_2} \int_0^{L_2} \int_0^{L_1} f(x, y) \sin \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2} dx dy. \tag{70}$$

EXAMPLE 9.15

Find the double Fourier series representation of $f(x, y) = xy$ over $-2 \le x \le 2$ and $-4 \le y \le 4$.

Solution The function $f(x, y)$ is odd in both x and y, so this corresponds to the double Fourier sine series representation of case (d) with $L_1 = 2$ and $L_2 = 4$. From (70) we have

$$\begin{aligned} d_{mn} &= \frac{4}{8} \int_0^4 \int_0^2 xy \sin \frac{m\pi x}{2} \sin \frac{n\pi y}{4} dx dy \\ &= \frac{1}{2} \left[\int_0^2 x \sin \frac{m\pi x}{2} dx \right] \left[\int_0^4 y \sin \frac{n\pi y}{4} dy \right] \\ &= \frac{1}{2} \left[\frac{-4(-1)^m}{m\pi} \right] \left[\frac{-16(-1)^n}{n\pi} \right] = (-1)^{m+n} \frac{32}{mn\pi^2}. \end{aligned}$$

Thus, the required double Fourier sine series representation is

$$f(x, y) = \frac{32}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} \frac{1}{mn} \sin \frac{m\pi x}{2} \sin \frac{n\pi y}{4},$$

for $-2 \leq x \leq 2$ and $-4 \leq y \leq 4$. Notice that this same expression describes the representation of $f(x, y)$ for $0 \leq x \leq 2$ and $0 \leq y \leq 4$. ■

By analogy with the half-range sine and cosine series of Section 9.3, a function $f(x, y)$ defined in a region $0 \leq x \leq a, 0 \leq y \leq b$ can be extended to the region $-a \leq x \leq a, -b \leq y \leq b$ either as a function that is odd in both x and y, or as one that is even in both x and y. If it is extended as an odd function, case (d) applies and the representation in the first quadrant follows by restricting the result to $0 \leq x \leq a$, $0 \leq y \leq b$, whereas if it is extended as an even function, case (a) applies, when the representation is again obtained by restricting the result to $0 \leq x \leq a, 0 \leq y \leq b$.

Suppose, for example, a double Fourier sine series representation of $f(x, y) = xy$ is required for $0 \leq x \leq 2$ and $0 \leq y \leq 4$. Then extending $f(x, y)$ to the region $-2 \leq x \leq 2, -4 \leq y \leq 4$ as a function that is odd in both x and y leads to Example 9.15, so the required representation is given by restricting the double Fourier sine series of Example 9.15 to $0 \leq x \leq 2$ and $0 \leq y \leq 4$. Similarly, $f(x, y) = xy$ can be represented by a double Fourier cosine series in $0 \leq x \leq 2$ and $0 \leq y \leq 4$ by extending it as $f(x, y) = |x||y|$ for $-2 \leq x \leq 2$ and $-4 \leq y \leq 4$. As $f(x, y)$ is even in both x and y, case (a) can be applied and the result again restricted so that $0 \leq x \leq 2$ and $0 \leq y \leq 4$.

A typical plot of a double Fourier series approximation to $f(x, y) = xy$ for $0 \leq x \leq 2$ and $0 \leq y \leq 4$ provided by a partial sum of the double Fourier sine series in Example 9.15 is shown in Fig. 9.14 for the case with $m = n = 10$. If, instead, the cosine approximation had been used (see Exercise 6), the plot of the corresponding approximation provided by the partial sum with $m = n = 10$ is shown in Fig. 9.15. The convergence of the double cosine series is seen to be the faster of the two.

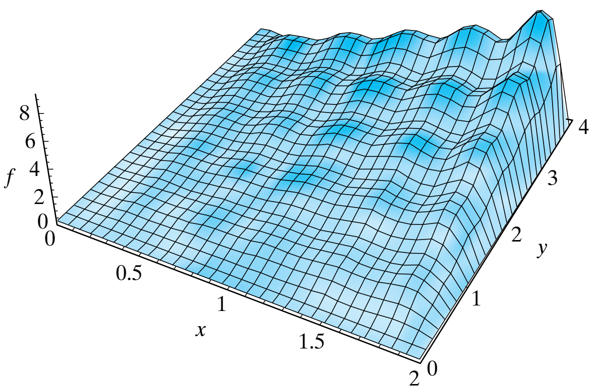


FIGURE 9.14 A double Fourier sine series approximation to $f(x, y) = xy$.

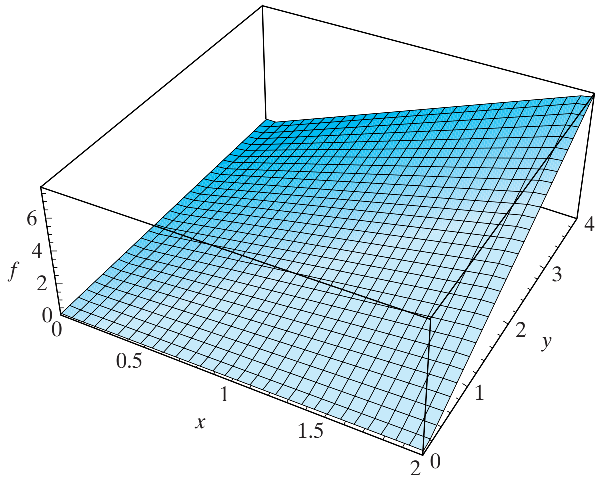


FIGURE 9.15 A double Fourier cosine series approximation to $f(x, y) = xy$.

Summary

It was shown how an ordinary Fourier series representation can be extended in a natural way to the expansion of functions $f(x, y)$ of two variables. After the derivation of the general expansion result, four useful special cases were examined and illustrated by example. Unless $f(x, y)$ is simple, the Fourier series approximation of functions of two variables can require numerical integration when finding the Fourier coefficients, and many terms are usually required to achieve good convergence, so in general it is necessary to perform such calculations and to plot the result by computer.

EXERCISES 9.6

1. By setting $y = 1$ in $f(x, y) = x^2 y$, with $-\pi \leq x \leq \pi$ and $-\pi \leq y \leq \pi$, show that the double Fourier series representation of $f(x, y)$ reduces to the ordinary Fourier series representation of $f(x) = x^2$ for $-\pi \leq x \leq \pi$ given by

$$f(x) = \frac{\pi^2}{3} + 4\sum_{m=1}^{\infty}(-1)^m \frac{\cos mx}{m^2}$$

In Exercises 2 through 9 find and plot double Fourier series partial sum approximations to the given function.

2. $f(x, y) = xy^2$, for $-\pi \leq x \leq \pi$ and $-\pi \leq y \leq \pi$.

3. $f(x, y) = x^3 y$, for $-\pi \leq x \leq \pi$ and $-\pi \leq y \leq \pi$.

4. $f(x, y) = x^2 y^2$, for $-\pi \leq x \leq \pi$ and $-\pi \leq y \leq \pi$.

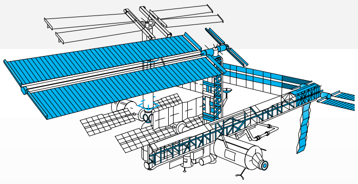
5.* $f(x, y) = \text{sign}(xy)$, for $-\pi \leq x \leq \pi$ and $-\pi \leq y \leq \pi$, where sign $u = 1$ if $u > 0$ and sign $u = -1$ if $u < 0$.

6.* $f(x, y) = |xy|$, for $-2 \leq x \leq 2$ and $-4 \leq y \leq 4$.

7.* $f(x, y) = \text{sign}\,(xy) + xy$, for $-\pi \leq x \leq \pi$ and $-\pi \leq y \leq \pi$.

8.* $f(x, y) = y|\sin x|$, for $-\pi \leq x \leq \pi$ and $-\pi \leq y \leq \pi$.

9.* Extend $f(x, y) = xy^2$, for $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$, to $-\pi \leq x \leq \pi$ and $-\pi \leq y \leq \pi$ as an odd function, and hence find a double Fourier sine series representation of $f(x, y)$ for $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$.



CHAPTER 9
TECHNOLOGY PROJECTS

The purpose of these projects is to use computer algebra to generate Fourier series for continuous and discontinuous functions, to use computer graphics to examine their convergence to the functions they represent, and to explore the nature of the Gibbs phenomenon.

Project 1

Finding Fourier Series and Plotting Partial Sums

Use computer algebra to find the first 11 terms $a_0, a_1, \ldots, a_5, b_1, b_2, \ldots, b_5$ of the Fourier series of

$$f(x) = (\pi^2 - x^2)e^{-x}\sin x \quad \text{for } -\pi \le x \le \pi.$$

Plot the approximation to $f(x)$ obtained by using (a) the terms involving a_0, a_1, a_2, b_1, and b_2 and (b) the 11 terms involving $a_0, \ldots, a_5, b_1, \ldots, b_5$ in the partial sum approximation, and compare the results with the graph of $f(x)$.

Project 2

Examining the Gibbs Phenomenon

Use computer algebra to find the Fourier series representation of the function

$$f(x) = \begin{cases} \sin x - 1, & -\pi < x < 0 \\ \sin x + 1, & 0 < x < \pi. \end{cases}$$

By plotting the partial sum representations of $f(x)$ using different numbers of terms, demonstrate the persistence of the overshoot and undershoot caused by the Gibbs phenomenon as the number of terms in the approximation increases.

Project 3

The Complex Fourier Series

Use computer algebra with the complex Fourier series representation of a function to verify the coefficients c_n and c_{2n-1} found in Example 9.12. Plot different partial sum approximations to $f(x)$ and, as in Project 2, demonstrate the persistence of the Gibbs phenomena as the number of terms in the partial sum approximation increases.

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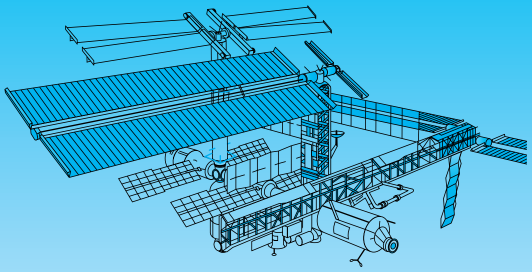
CHAPTER 10



Fourier Integrals and the Fourier Transform

Fourier series enable functions and solutions of linear systems defined over a finite interval to be represented as an infinite series of sines and cosines. This suffices for many physical problems, but often the interval involved is either semi-infinite or infinite, in which case a somewhat different representation becomes necessary. This happens, for example, when working with the partial differential equations that describe heat conduction and diffusion in a half-space for which Fourier series cannot be used.

The Fourier integral can be regarded as the limiting case of a Fourier series representation of a function $f(x)$ defined over an interval $-L < x < L$ as $L \to \infty$. The meaning of the integral representation when the function to be represented is discontinuous is considered, and the special cases of the sine and cosine integral representations are introduced.

Fourier sine and cosine transforms are considered, tables of their transform pairs are given, and the transform of derivatives is discussed. In anticipation of Chapter 18, an application of the Fourier transform is made to the problem of the one-dimensional time dependent heat equation.

10.1 The Fourier Integral

A Fourier series has been shown to represent an arbitrary function $f(x)$ over an interval $-L \le x \le L$, and because the series is periodic with period $2L$ the representation of $f(x)$ in this fundamental interval is repeated by periodicity for all x outside the interval. However, even if $f(x)$ is defined outside the fundamental interval, it does not necessarily follow that the function and its periodic extensions coincide outside the interval. This means that if a nonperiodic function is to be represented over an arbitrarily large interval, some generalization of a Fourier series is required.

Letting $L \to \infty$ in a Fourier series leads to the introduction of a different type of representation called a **Fourier integral representation**, where the function $f(x)$ is defined for all x and need not be periodic. This representation forms the basis of an integral transform called the **Fourier transform** that is similar to the Laplace transform. As with the Laplace transform, one of the the main uses of the Fourier transform is in the solution of differential equations.

The derivation of the Fourier integral representation given here is heuristic, because a rigorous one requires techniques that are not needed elsewhere in the book. We start from the definition of a Fourier series of $f(x)$ over an interval $-L \le x \le L$ given in (18) and (19) of Section 9.1 by writing

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{1}$$

where

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx \quad \text{for } n = 1, 2, \dots. \end{aligned} \tag{2}$$

Substituting the Fourier coefficients (2) into Fourier series (1) allows it to be written in the integral form

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(u)du + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(u)\cos\frac{n\pi(u-x)}{L}du. \tag{3}$$

To proceed further, if the representation is to remain valid as $L \to \infty$ the first term must not become either infinite or indeterminate. This will certainly be true if $\lim_{L\to\infty}\int_{-L}^{L}|f(x)|dx$ is finite, because then the integral involved in the first term will be *absolutely convergent* and the first term in (3) will vanish in the limit as $L \to \infty$. From now on we will assume this condition to be satisfied. We can now write (3) as

$$f(x) = \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(u)\cos\frac{n\pi(u-x)}{L}du. \tag{4}$$

It is from this point onward that our derivation of the Fourier integral representation becomes heuristic, because the arguments used to convert (4) to an integral over the interval $(-\infty, \infty)$ are merely intuitive. A careful examination of the convergence of the double integral involved would be necessary to provide a rigorous justification.

Setting $\Delta_n\omega = \pi/L$, and defining the frequency $\omega_n = n\pi/L$, allows (4) to be rewritten as

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\Delta_n\omega\int_{-L}^{L} f(u)\cos[\omega_n(u-x)]du. \tag{5}$$

Examination of (5) suggests it is equivalent to the pre-limit sum approximation used in the definition of the definite (Riemann) integral of the function

$$F(u) = \frac{1}{\pi}\int_{-L}^{L} f(u)\cos\omega(u-x)du.$$

Using this last result in (5), and proceeding to the limit as $L \to \infty$, we obtain

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty} d\omega\int_{-\infty}^{\infty} f(u)\cos\omega(u-x)du, \tag{6}$$

the Fourier integral representation

which is called the **Fourier integral representation** of $f(x)$.

By defining the functions $A(\omega)$ and $B(\omega)$ as

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\omega u du \quad \text{and} \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\omega u du, \tag{7}$$

the Fourier integral representation in (6) can be written in the simpler form

$$f(x) = \int_{0}^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega. \tag{8}$$

The convergence properties of Fourier series recorded in Theorem 9.1 can be shown to be transferred to the Fourier integral representation of $f(x)$ if, in addition to the integral of $f(x)$ being absolutely convergent over $(-\infty, \infty)$, it also satisfies certain other conditions. These conditions, called **Dirichlet conditions**, are as follows:

Dirichlet conditions

(i) In any finite interval $f(x)$ has only a finite number of maxima and minima

(ii) In any finite interval $f(x)$ has only a finite number of bounded jump discontinuities and no infinite jump discontinuities.

We now state the following theorem for the Fourier integral without proof.

PETER GUSTAV LEJEUNE DIRICHLET (1805–1859)
A German mathematician who studied under Gauss, was the son-in-law of Jacobi and succeeded Gauss as Professor of Mathematics at Göttingen. He did much to make some of the more abstruse contributions by Gauss better understood. His most important contributions to mathematics were his major contribution to the understanding of the convergence of Fourier series, and his work on number theory and the theory of potential.

THEOREM 10.1

the fundamental Fourier integral theorem

Fourier integral theorem Let $f(x)$ satisfy Dirichlet conditions, and suppose the (sufficiency) conditions that $f(x)$ be both integrable and absolutely integrable over the interval $-\infty < x < \infty$ are both satisfied, so each of the integrals $\int_{-\infty}^{\infty} f(x)dx$ and $\int_{-\infty}^{\infty} |f(x)|dx$ exists. Then

$$\frac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{\pi}\int_{0}^{\infty} d\omega \int_{-\infty}^{\infty} f(u)\cos\omega(u-x)du$$

or, equivalently,

$$\frac{1}{2}[f(x+0) + f(x-0)] = \int_{0}^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega,$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\omega u du \quad \text{and} \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\omega u du.$$ ■

EXAMPLE 10.1

Find the Fourier integral representation of $f(x) = e^{-|x|}$.

Solution The function $e^{-|x|}$ satisfies the Dirichlet conditions, and $\int_{-\infty}^{\infty} |e^{-|x|}|dx = 2$, so the integral of $f(x) = e^{-|x|}$ over $(-\infty, \infty)$ is absolutely convergent. This confirms that $f(x) = e^{-|x|}$ has a Fourier integral representation.

The function $e^{-|x|}$ is even in x, so $e^{-|u|}\cos\omega u$ is also even, and

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{-|u|}\cos\omega u du=\frac{2}{\pi}\int_{0}^{\infty}e^{-u}\cos\omega u du=\frac{2}{\pi(1+\omega^2)}.$$

As the function $e^{-|u|}\sin\omega u$ is odd in u,

$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{-|u|}\sin\omega u du=\frac{2}{\pi}\int_{0}^{\infty}e^{-u}\sin\omega u du=0,$$

so from (8) the Fourier integral representation of $e^{-|x|}$ is seen to be

$$e^{-|x|}=\frac{2}{\pi}\int_{0}^{\infty}\frac{\cos\omega x}{1+\omega^2}d\omega.$$

■

EXAMPLE 10.2 Find the Fourier integral representation of

$$f(x)=\begin{cases}e^{-x}, & x>0\\ 0, & x<0\end{cases}$$

and use Theorem 10.1 to find the value of the resulting integral when (a) $x<0$, (b) $x=0$, and (c) $x>0$.

Solution The function $f(x)$ satisfies the Dirichlet conditions and the integral $\int_{-\infty}^{\infty}|f(x)|dx=\int_{0}^{\infty}e^{-x}dx=1$, so as the conditions of Theorem 10.1 are satisfied the function has a Fourier integral representation.

We have

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\cos\omega u du=\frac{1}{\pi}\int_{0}^{\infty}e^{-u}\cos\omega u du=\frac{1}{\pi(1+\omega^2)}$$

and

$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\sin\omega u du=\frac{1}{\pi}\int_{0}^{\infty}e^{-u}\sin\omega u du=\frac{\omega}{\pi(1+\omega^2)}.$$

Substituting into (8) shows the Fourier integral representation to be

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\frac{\cos\omega x+\omega\sin\omega x}{1+\omega^2}d\omega\quad\text{for } -\infty<x<\infty.$$

Applying the results of Theorem 10.1 to this integral, we find that

$$\pi f(x)=\int_{0}^{\infty}\frac{\cos\omega x+\omega\sin\omega x}{1+\omega^2}d\omega=\begin{cases}0, & x<0\\ \pi/2, & x=0\\ \pi e^{-x}, & x>0.\end{cases}$$

When $x=0$, this last result is seen to reduce to the familiar definite integral

$$\int_{0}^{\infty}\frac{d\omega}{1+\omega^2}=\frac{\pi}{2}.$$

■

Special forms of the Fourier integral representation arise according to whether $f(x)$ is even or odd. When $f(x)$ is an even function, $f(u)\sin\omega u$ is an odd function of u,

so $B(\omega) \equiv 0$ and

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(u)\cos\omega u\,du, \tag{9}$$

so that (8) simplifies to the **Fourier cosine integral representation** of $f(x)$

$$f(x) = \int_0^\infty A(\omega)\cos\omega x\,d\omega. \tag{10}$$

Similarly, when $f(x)$ is an odd function, $f(u)\cos\omega u$ is an odd function of u, so $A(\omega) \equiv 0$ and

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(u)\sin\omega u\,du, \tag{11}$$

causing (8) to simplify to the **Fourier sine integral representation** of $f(x)$ given by

$$f(x) = \int_0^\infty B(\omega)\sin\omega x\,d\omega. \tag{12}$$

different Fourier integral representations

Summary of Fourier integral representations

(a) An *arbitrary* function $f(x)$ satisfying the conditions of Theorem 10.1 has the **general Fourier integral representation**

$$\frac{1}{2}[f(x+0)+f(x-0)] = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega. \tag{13}$$

(b) An *even* function $f(x)$ satisfying the conditions of Theorem 10.1 has the **Fourier cosine integral representation**

$$\frac{1}{2}[f(x+0)+f(x-0)] = \int_0^\infty A(\omega)\cos\omega x\,d\omega. \tag{14}$$

(c) An *odd* function $f(x)$ satisfying the conditions of Theorem 10.1 has the **Fourier sine integral representation**

$$\frac{1}{2}[f(x+0)+f(x-0)] = \int_0^\infty B(\omega)\sin\omega x\,d\omega, \tag{15}$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(u)\cos\omega u\,du \quad \text{and} \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(u)\sin\omega u\,du. \tag{16}$$

Summary

The Fourier integral representation of a function $f(x)$ was introduced as the natural extension of a Fourier series representation as the interval of the representation extends to become the interval $-\infty < x < \infty$. A fundamental representation theorem was given and illustrated by example, and some useful special cases of the theorem were considered.

EXERCISES 10.1

Find the Fourier integral representation of the given functions.

1. The rectangular pulse function $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ (Fig. 10.1).

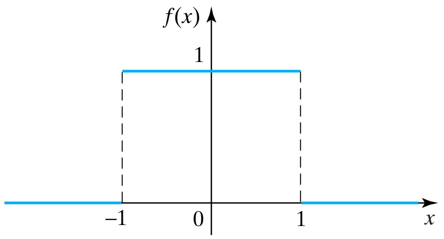


FIGURE 10.1 The rectangular pulse function.

2. The triangular function

$$f(x) = \begin{cases} 0, & |x| > a \\ b\left(1 + \dfrac{x}{a}\right), & -a \le x \le 0 \\ b\left(1 - \dfrac{x}{a}\right), & 0 \le x \le a \end{cases} \quad \text{(Fig. 10.2).}$$

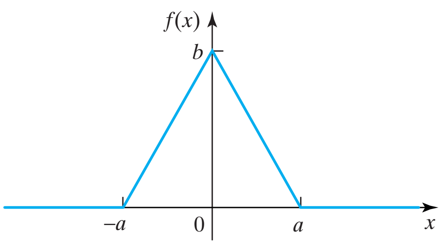


FIGURE 10.2 The triangular function.

3. $f(x) = \begin{cases} 0, & |x| > a \\ bx/a, & -a \le x \le a \end{cases}$ (Fig. 10.3).

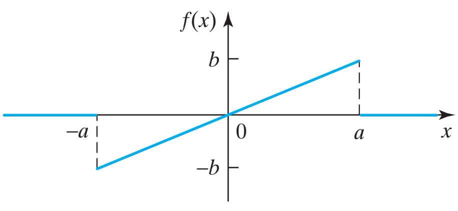


FIGURE 10.3 The truncated straight line function.

4. $f(x) = \begin{cases} 0, & x \le 0 \\ \sin x, & 0 \le x \le \pi \\ 0, & x \ge \pi \end{cases}$ (Fig. 10.4).

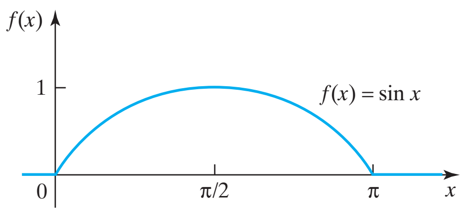


FIGURE 10.4 The asymmetric truncated sine function.

5. $f(x) = \begin{cases} (\pi/2)\cos x, & |x| < \pi/2 \\ 0, & |x| > \pi/2 \end{cases}$ (Fig. 10.5).

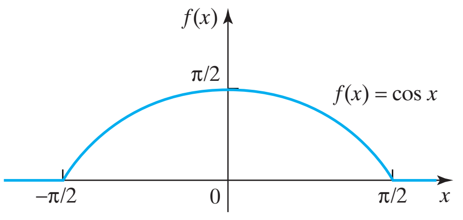


FIGURE 10.5 The truncated cosine function.

6. $f(x) = \begin{cases} (\pi/2)\sin x, & |x| < \pi/2 \\ 0, & |x| > \pi/2 \end{cases}$ (Fig. 10.6).

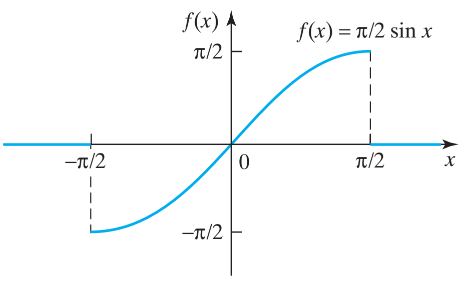


FIGURE 10.6 The truncated sine function.

7. $f(x) = \begin{cases} 0, & x < 0 \\ \cos x, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$ (Fig. 10.7).

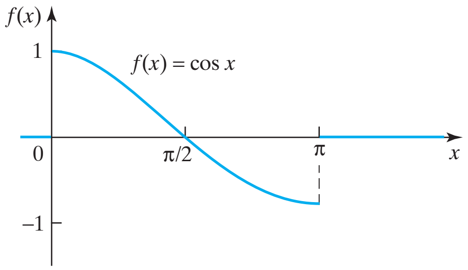


FIGURE 10.7 The asymmetric truncated cosine function.

8. The hump function $f(x) = 1/(1 + x^2)$ (Fig. 10.8). (Hint: Use the result of Example 10.16 with a change of notation.)

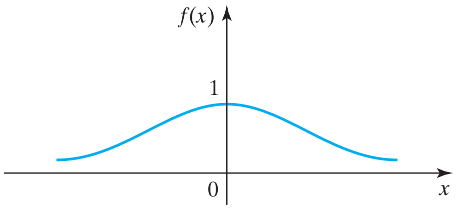


FIGURE 10.8 The hump function.

10.2 The Fourier Transform

The starting point for the development of the *Fourier transform* is the complex form of the Fourier integral representation of a function $f(x)$. To derive this representation in which $f(x)$ is defined over the interval $(-\infty, \infty)$, we substitute into (8) of Section 10.1 the expressions for $A(\omega)$ and $B(\omega)$ given in (7) to obtain

$$\begin{aligned}\frac{1}{2}[f(x+0) + f(x-0)] &= \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(u)[\cos\omega u\cos\omega x + \sin\omega u\sin\omega x]du\right]d\omega \\ &= \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(u)\cos\{\omega(u-x)\}du\right]d\omega \\ &= \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(u)\cos\{\omega(x-u)\}du\right]d\omega,\end{aligned}$$

where we have used the result $\cos\omega(u-x) = \cos\omega(x-u)$.

As the integrand in the last integral is an even function of ω, the interval of integration with respect to ω can be doubled and the result compensated by the introduction of a multiplicative factor 1/2 to give

$$\frac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\cos\omega(x-u)du\right]d\omega. \tag{17}$$

The function $\sin\omega(x-u)$ is an odd function of ω, so it follows directly that

$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\sin\{\omega(x-u)\}du\right]d\omega. \tag{18}$$

the complex Fourier integral representation

Multiplying equation (18) by i, adding the result to equation (17), and using the Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$, we arrive at the **complex Fourier integral**

representation

$$\frac{1}{2}[f(x+0)+f(x-0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\exp\{i\omega(x-u)\}du\right]d\omega. \tag{19}$$

The brackets in (17) to (19) were retained to clarify the order in which the integrations are performed, but they are usually omitted in (19), which then becomes

$$\frac{1}{2}[f(x+0)+f(x-0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\exp\{i\omega(x-u)\}du\,d\omega. \tag{20}$$

Clearly, the left-hand side of (20) reduces to $f(x)$ wherever the function is continuous.

To arrive at the definitions of a Fourier transform and its inverse we write the factor $\exp\{i\omega(x-u)\}$ in (19) (equivalently (20)) as the product $\exp\{i\omega x\}\cdot\exp\{-i\omega u\}$. Then, as the inner integral only involves integration with respect to u, we rewrite (19) as

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\{i\omega x\}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\exp\{-i\omega u\}du\right]d\omega, \tag{21}$$

where the left-hand side is to be replaced by $(1/2)[f(x+0)+f(x-0)]$ whenever $f(x)$ is discontinuous.

If we now define the function $F(\omega)$ as

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\exp\{-i\omega u\}du,$$

then because u is a dummy variable it can be replaced by x and the result rewritten as

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\exp\{-i\omega x\}dx, \tag{22}$$

so that (19) becomes

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)\exp\{i\omega x\}d\omega. \tag{23}$$

Fourier transforms and transform pairs

The function $F(\omega)$ in (22) is called the **Fourier transform** of $f(x)$, or sometimes the **exponential Fourier transform**, and because integral (23) recovers $f(x)$ from $F(\omega)$ it is called the **inversion integral** for the Fourier transform. As with the Laplace transform, when working with the Fourier transform the function $f(x)$ and the associated Fourier transform $F(\omega)$ are called a **Fourier transform pair**. A short table of Fourier transform pairs is to be found at the end of this section.

Various other notations are used to indicate the Fourier transform of $f(x)$, the most common of which involves representing it by $\hat{f}(\omega)$, so in terms of the notation used here, $\hat{f}(\omega) = F(\omega)$.

Another notation that is often useful involves representing the Fourier transform of $f(x)$ by $\mathcal{F}\{f(x)\}$, so that $\mathcal{F}\{f(x)\} = F(\omega)$, and when this notation is used the inverse Fourier transform is written $\mathcal{F}^{-1}\{F(\omega)\} = f(x)$. In what follows a function to be transformed is denoted by a lowercase letter, and the corresponding uppercase letter is then used to denote its Fourier transform. So, for example, $\mathcal{F}\{g(x)\} = G(\omega)$ and $\mathcal{F}\{h(x)\} = H(\omega)$.

The choice of the normalizing factors $1/\sqrt{2\pi}$ in integrals (22) and (23) is optional, and it is chosen here to introduce as much symmetry as possible into the definitions of a Fourier transform and its inverse. All that is required of the normalizing factors is that their product be $1/(2\pi)$, so in many reference works the factor $1/\sqrt{2\pi}$ in (22) is replaced by 1, while the factor $1/\sqrt{2\pi}$ in (23) is replaced by $1/(2\pi)$. It is impossible to achieve complete symmetry in the definitions of a Fourier integral and its inverse because the exponential factor occurs with opposite signs in (22) and (23).

When Fourier transforms listed in reference works are used, another source of confusion can arise because sometimes the signs in the exponential factors occurring in integrals (22) and (23) are interchanged. When this happens a Fourier transform obtained using this sign convention can be converted to the one used here by reversing the sign of ω. However, each definition of the Fourier transform and the corresponding inversion integral conform to the general pattern

$$\mathcal{F}\{f(x)\} = \frac{k}{2\pi}\int_{-\infty}^{\infty} f(x)\exp\{\pm i\omega x\}dx \quad \text{and}$$
$$\mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{k}\int_{-\infty}^{\infty} F(\omega)\exp\{\mp i\omega x\}d\omega, \tag{24}$$

where k is an arbitrary scale factor.

In view of the different conventions that are in use, when working with Fourier transforms and referring to reference works, it is essential that the normalizing factor k and the sign convention employed in the exponential factors be established before any use is made of the results.

When we considered the convergence of Fourier series, the Riemann–Lebesgue lemma was established the results of which were that

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\,\cos nx dx = \lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\,\sin nx dx = 0. \tag{25}$$

A limiting argument similar to the one used in Section 10.1 when deriving the Fourier integral representation of $f(x)$ shows that, provided $f(x)$ has a Fourier transform,

$$\lim_{|\omega|\to\infty}\int_{-\infty}^{\infty} f(x)\,\cos\omega x dx = \lim_{|\omega|\to\infty}\int_{-\infty}^{\infty} f(x)\,\sin\omega x dx = 0. \tag{26}$$

As the Fourier transform $F(\omega)$ of $f(x)$ can be written

$$F(\omega) = \tfrac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} f(x)\,\cos\omega x dx - i\int_{-\infty}^{\infty} f(x)\,\sin\omega x dx\right], \tag{27}$$

an application of limits (26) in (27) establishes the important property of a Fourier transform that

$$\lim_{|\omega|\to\infty} F(\omega) = 0. \tag{28}$$

EXAMPLE 10.3

Find the Fourier transforms of

(a) $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a, \end{cases}$ (b) $g(x) = \begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise}, \end{cases}$ (c) $p(x) = \dfrac{1}{x^2 + a^2}$

by making use of the standard integral $\int_{-\infty}^{\infty} \frac{\cos \omega x}{x^2+a^2} dx = \frac{\pi}{a} e^{-|\omega|a}$ $(a > 0)$ and (d) $q(x) = \begin{cases} e^{iax}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$. In each case confirm that the Fourier transform vanishes as $\omega \to \pm\infty$.

Solution

(a)

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-i\omega x} dx = \frac{1}{\omega\sqrt{2\pi}} \left[\frac{e^{i\omega a} - e^{-i\omega a}}{i} \right]$$

$$= \frac{1}{\omega}\sqrt{\frac{2}{\pi}} \left[\frac{e^{i\omega a} - e^{-i\omega a}}{2i} \right] = \sqrt{\frac{2}{\pi}} \frac{\sin \omega a}{\omega}.$$

As $\sin \omega a$ is bounded, it follows directly that $\lim_{|\omega|\to\infty} F(\omega) = 0$.

(b)

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{0}^{a} e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \left(\frac{1 - e^{-i\omega a}}{i\omega} \right).$$

As the numerator of $G(\omega)$ is bounded, it follows that $\lim_{|\omega|\to\infty} G(\omega) = 0$. This example shows that although $f(x)$ may be real, its Fourier transform can be complex.

(c)

$$P(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{x^2 + a^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\cos \omega x}{x^2 + a^2} dx - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sin \omega x}{x^2 + a^2} dx.$$

The integrand of the second integral is odd, so the value of the integral is zero. Using the standard result

$$\int_{-\infty}^{\infty} \frac{\cos \omega x}{x^2 + a^2} dx = \frac{\pi}{a} e^{-|\omega|a}$$

in the remaining integral on the right, we find that

$$P(\omega) = \sqrt{\frac{\pi}{2}} \frac{e^{-|\omega|a}}{a} \quad (a > 0).$$

In this case the factor $e^{-|\omega|a}$ ensures that $\lim_{|\omega|\to\infty} P(\omega) = 0$.

(d)

$$Q(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{1} e^{-i(\omega - a)x} dx$$

$$= \frac{i}{\sqrt{2\pi}} \left(\frac{1 - e^{-i(\omega - a)}}{a - \omega} \right).$$

As the numerator of the Fourier transform is bounded, the denominator causes the transform to vanish as $|\omega| \to \infty$. This example shows that a complex function can also have a Fourier transform and, in general, that the transform will be complex. ■

the main operational properties of Fourier transforms

The fundamental properties contained in Theorems 10.2 to 10.8 that follow are called **operational properties** of the Fourier transform. Familiarity with these properties is essential, because they simplify calculations involving Fourier transforms and can lead to results that are difficult to obtain without their use.

THEOREM 10.2

Linearity of the Fourier transform Let the functions $f(x)$ and $g(x)$ have the respective Fourier transforms $F(\omega)$ and $G(\omega)$, and let a and b be arbitrary constants. Then

$$\mathcal{F}\{af(x)+bg(x)\} = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}.$$

Proof As the Fourier integral involves the operation of integration, the linearity property of the transform follows directly from the linearity property of the definite integral. ■

Theorem 10.2 is important when the Fourier transform of a sum of functions is required, because it is this result that allows each term involved in the sum to be transformed separately before the results are added.

EXAMPLE 10.4

Find the Fourier transform of $3f(x) - 2g(x)$, where $f(x)$ and $g(x)$ are the functions in (a) and (b) of Example 10.3.

Solution Using the results of Example 10.3 and applying Theorem 10.2, we have

$$\begin{aligned}\mathcal{F}\{3f(x)-2g(x)\} &= 3\mathcal{F}\{f(x)\} - 2\mathcal{F}\{g(x)\}\\ &= \sqrt{\frac{2}{\pi}}\left\{\frac{3\sin\omega a}{\omega} - \left(\frac{1-e^{-i\omega a}}{i\omega}\right)\right\}.\end{aligned}$$ ■

THEOREM 10.3

Fourier transform of a derivative of $f(x)$ Let $f(x)$ be a continuous function of x with the property that $\lim_{|x|\to\infty} f(x) = 0$, and such that $f'(x)$ is absolutely integrable over $(-\infty, \infty)$. Then:

(a) $\mathcal{F}\{f'(x)\} = i\omega F(\omega)$.

(b) For all n such that the derivatives $f^{(r)}(x)$ with $r = 1, 2, \ldots, n$ satisfy Dirichlet conditions, are absolutely integrable over $(-\infty, \infty)$, and $\lim_{|x|\to\infty} f^{(n-1)}(x) = 0$,

$$\mathcal{F}\{f^{(n)}(x)\} = (i\omega)^n F(\omega),$$

where $f^{(n)}(x) = d^n f/dx^n$.

Proof

(a) Integration by parts coupled with the condition that $\lim_{|x|\to\infty} f(x) = 0$ gives

$$\begin{aligned}\mathcal{F}\{f'(x)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{-i\omega x}dx\\ &= \frac{1}{\sqrt{2\pi}}\left[f(x)e^{-i\omega x}\Big|_{-\infty}^{\infty} - (-i\omega)\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx\right]\\ &= i\omega\ \mathcal{F}\{f(x)\} = i\omega F(\omega),\end{aligned}$$

where the term $f(x)e^{-i\omega x}|_{-\infty}^{\infty}$ vanishes because of the condition $\lim_{|x|\to\infty} f(x) = 0$.

(b) The second part of the theorem follows by repeated application of result (a), and the conditions imposed on $f^{(n)}(x)$ are necessary to ensure that its Fourier transform exists. ■

EXAMPLE 10.5 Find the Fourier transform of $p'(x)$ from the Fourier transform of $p(x)$, where $p(x)$ is the function in Example 10.3(c).

Solution It was shown in Example 10.3(c) that $P(\omega) = \sqrt{\frac{\pi}{2}}\frac{e^{-|\omega|a}}{a}$, so it follows from Theorem 10.3 (a) that $\mathcal{F}\{p'(x)\} = i\omega P(\omega) = i\omega\sqrt{\frac{\pi}{2}}\frac{e^{-|\omega|a}}{a}$. ■

THEOREM 10.4 **Fourier transform of $x^n f(x)$** Let $f(x)$ be a continuous and differentiable function with an n times differentiable Fourier transform $F(\omega)$. Then

(a) $$\mathcal{F}\{xf(x)\} = i\frac{d}{d\omega}[F(\omega)]$$

and

(b) $$\mathcal{F}\{x^n f(x)\} = i^n\frac{d^n}{d\omega^n}[F(\omega)],$$

for all n such that $\lim_{|\omega|\to\infty} F^{(n)}(\omega) = 0$.

Proof The proof of the theorem follows directly by the application of *Leibniz's rule* that governs differentiation under the integral sign. The rule may be stated as follows:

Leibniz' rule: Let $f(x,\omega)$ and $\partial f/\partial\omega$ be continuous functions of their variables with $-\infty < x < \infty$ and $-\infty < \omega < \infty$. Furthermore, let $\int_{-\infty}^{\infty}|f(x,\omega)|dx$ be finite and $|\partial f/\partial\omega| \le h(x)$ where $h(x)$ is piecewise continuous and such that $\int_{-\infty}^{\infty} h(x)dx$ is finite. Then

$$\frac{d}{d\omega}\int_{-\infty}^{\infty} f(x,\omega)dx = \int_{-\infty}^{\infty}\frac{\partial}{\partial\omega}[f(x,\omega)]dx.$$

(a) Using Leibniz' rule to differentiate the Fourier transform of $f(x)$, we obtain

$$\frac{d}{d\omega}[F(\omega)] = \frac{1}{\sqrt{2\pi}}\frac{d}{d\omega}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xf(x)e^{-i\omega x}dx.$$

The required result follows from this after multiplication by i, because the expression on the right is then $\mathcal{F}\{xf(x)\}$.

(b) The proof for the case when $n > 1$ follows by repeated application of result (a). The conditions imposed on $x^n f(x)$ and $F(\omega)$ are necessary to ensure the existence of the Fourier transform. ■

THEOREM 10.5 **Fourier transform of $x^m f^{(n)}(x)$** Let $f(x)$ be a continuous n times differentiable function. Furthermore, let $x^m f^{(r)}(x)$ for $r = 1, 2, \ldots, n$ satisfy Dirichlet conditions and be absolutely integrable over $(-\infty, \infty)$, and let $\omega^n F(\omega)$ possess an m times differentiable inverse Fourier transform. Then, provided $\lim_{|x|\to\infty} f^{(n-1)}(x) = 0$,

$$\mathcal{F}\{x^m f^{(n)}(x)\} = (i)^{m+n}\frac{d^m}{d\omega^m}[\omega^n F(\omega)].$$

Proof The result follows directly by combining Theorems 10.3 and 10.4, because

$$\mathcal{F}\{x^m f^{(n)}(x)\} = (i)^m \frac{d^m}{d\omega^m}\mathcal{F}\{f^{(n)}(x)\} = (i)^{m+n}\frac{d^m}{d\omega^m}[\omega^n F(\omega)].$$

The conditions imposed on $x^m f^{(n)}(x)$ and $\omega^n F(\omega)$ are necessary to ensure the existence of the Fourier transform. ■

The examples that follow illustrate how Theorems 10.3 to 10.5 may be used to find the Fourier transforms of more complicated functions.

EXAMPLE 10.6 Find the Fourier transform of $f(x) = \exp(-a^2x^2)(a > 0)$.

Solution The function $f(x)$ is continuous and differentiable for all x and

$$\int_{-\infty}^{\infty} |\exp(-a^2x^2)|dx = \int_{-\infty}^{\infty} \exp(-a^2x^2)dx = \frac{1}{a}\int_{-\infty}^{\infty}\exp(-u^2)du = \frac{\sqrt{\pi}}{a},$$

where we have made use of the standard integral $\int_{-\infty}^{\infty}\exp(-u^2)du = \sqrt{\pi}$. This shows that $f(x)$ is absolutely integrable over the interval $(-\infty,\infty)$, and so $f(x)$ has a Fourier transform. A straightforward calculation establishes that $f(x)$ satisfies the differential equation

$$f' + 2a^2xf = 0.$$

Taking the Fourier transform of this equation using Theorem 10.2 gives

$$\mathcal{F}\{f'(x)\} + 2a^2\mathcal{F}\{xf(x)\} = 0.$$

Applying Theorem 10.3 to the first term and Theorem 10.4 to the second term and cancelling a factor i reduces this to the variables separable equation for $F(\omega)$,

$$2a^2F' + \omega F = 0, \quad \text{where } F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-a^2x^2)e^{-i\omega x}dx.$$

When variables are separated, the equation becomes

$$\int \frac{F'}{F}d\omega = -\frac{1}{2a^2}\int \omega d\omega,$$

so

$$\ln F(\omega) = -\frac{\omega^2}{4a^2} + \ln A, \quad \text{or} \quad F(\omega) = A\exp\left[-\frac{\omega^2}{4a^2}\right],$$

where, for convenience, the arbitrary integration constant has been written in the form $\ln A$. To determine A we use the fact that $A = F(0)$, but

$$F(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-a^2x^2)dx = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{\pi}}{a} = \frac{1}{a\sqrt{2}},$$

and so

$$\mathcal{F}\{\exp(-a^2x^2)\} = F(\omega) = \frac{1}{a\sqrt{2}}\exp\left\{-\frac{\omega^2}{4a^2}\right\} \quad (a > 0).$$ ■

EXAMPLE 10.7

finding the Fourier transform of a function defined by a differential equation

Find the Fourier transform of the Bessel function $J_0(x)$.

Solution The Bessel function $J_0(x)$ does not satisfy the absolute integrability condition found in Theorem 10.1. However, this is merely a sufficient condition that ensures the existence of the Fourier transform of a function $f(x)$, though not a necessary one. Functions exist that possess a Fourier transform even though this condition is violated, and $J_0(x)$ is such a function. The function $f(x) = J_0(x)$ is an even function that is defined for all x and satisfies Bessel's differential equation of order zero

$$xf'' + f' + xf = 0.$$

Taking the Fourier transform of the differential equation by using Theorem 10.2 and then applying Theorem 10.5 to the first term, Theorem 10.3 to the second term, and Theorem 10.4 to the last term, we find, after the cancellation of a factor i and the combination of terms, that

$$(1-\omega^2)F' - \omega F = 0, \quad \text{where } F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} J_0(x)e^{-i\omega x}dx.$$

This is a linear first order variables separable differential equation that can be written

$$\int \frac{F'}{F}d\omega = \int \frac{\omega}{1-\omega^2}d\omega,$$

so integration gives

$$\ln F(\omega) = -\frac{1}{2}\ln(1-\omega^2) + \ln A, \quad \text{or} \quad F(\omega) = \frac{A}{(1-\omega^2)^{1/2}}, \quad \text{with } 0 < \omega^2 < 1.$$

In this equation, the arbitrary integration constant has again been written in the form $\ln A$, and the restriction on ω^2 is necessary because the real logarithmic function is not defined for negative arguments.

To determine A we use the fact that $A = F(0)$, together with the standard result $\int_0^{\infty} J_0(x)dx = 1$ and the fact that $J_0(x)$ is an even function, to obtain

$$A = F(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} J_0(x)dx = \frac{2}{\sqrt{2\pi}}\int_0^{\infty} J_0(x)dx = \sqrt{\frac{2}{\pi}}.$$

Substituting A into $F(\omega)$ gives

$$\mathcal{F}\{J_0(x)\} = F(\omega) = \sqrt{\frac{2}{\pi}}\frac{1}{(1-\omega^2)^{1/2}}H(1-|\omega|),$$

where the Heaviside unit step function $H(1-|\omega|)$ is necessary because of the restriction imposed by the real logarithmic function that requires ω to be such that $0 < \omega^2 < 1$. ■

When working with Fourier integrals, as with the Laplace transform, it is useful to introduce the convolution operation to establish the relationship between the functions $f(x)$ and $g(x)$ and their respective Fourier transforms $F(\omega)$ and $G(\omega)$.

The **convolution** of functions $f(x)$ and $g(x)$ denoted by $f * g$ is a function of x, and if the dependence on a variable x in the convolution is to be emphasized,

it is then denoted by $(f * g)(x)$. The convolution of $f(x)$ and $g(x)$ is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt = \int_{-\infty}^{\infty} f(x-t)g(t)dt. \tag{29}$$

A slightly different definition of the convolution operation for the Fourier transform is also to be found in the literature, where it is defined as

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)g(x-t)dt.$$

When this definition is employed, the form taken by the next theorem (the convolution theorem for Fourier transforms) will require modification. This is because its form will depend on the factor $1/\sqrt{2\pi}$ and the way the constant 2π enters in the definition of the Fourier transform that is used.

THEOREM 10.6

relating the convolution of $f(x)$ and $g(x)$ and the product of their transforms

The convolution theorem for Fourier transforms Let the functions $f(x)$ and $g(x)$ be piecewise continuous, bounded, and absolutely integrable over $(-\infty, \infty)$ with the respective Fourier transforms $F(\omega)$ and $G(\omega)$. Then

(a) $$\mathcal{F}\{(f * g)(x)\} = 2\pi\, \mathcal{F}\{f(x)\}\mathcal{F}\{g(x)\}, \text{ or } \mathcal{F}\{f * g\} = 2\pi\, F(\omega)G(\omega)$$

and, conversely,

(b) $$(f * g)(x) = \sqrt{2\pi} \int_{-\infty}^{\infty} F(\omega)G(\omega)e^{i\omega x}d\omega.$$

Proof **(a)** By definition,

$$\begin{aligned}\mathcal{F}\{(f * g)(x)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)g(x-t)e^{-i\omega x}dt\right]dx \\ &= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)g(x-t)e^{-i\omega x}dx\right]dt,\right.\end{aligned}$$

where the second result follows from the first by a change in the order of integration. If we set $v = x - t$, this becomes

$$\begin{aligned}\mathcal{F}\{(f * g)(x)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[f(t)g(v)e^{-i\omega(t+v)}dt\right]dv \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \int_{-\infty}^{\infty} g(v)e^{-i\omega v}dv.\end{aligned}$$

However, t and v are dummy variables and so may be replaced by x, causing the preceding result to become

$$\mathcal{F}\{(f * g)(x)\} = \mathcal{F}\{f(x)\}2\pi\mathcal{F}\{g(x)\},$$

showing that

$$\mathcal{F}\{(f * g)(x)\} = 2\pi\, \mathcal{F}\{f(x)\}\, \mathcal{F}\{g(x)\}, \quad \text{or} \quad \mathcal{F}\{(f * g)(x)\} = 2\pi\, F(\omega)G(\omega).$$

Result (b) follows directly from the last result by taking the inverse Fourier transform that causes a factor $\sqrt{2\pi}$ to cancel. ■

EXAMPLE 10.8

It was shown in Example 10.3(a) that the function $f(x)=\begin{cases}1, & |x|<a\\ 0, & |x|>a\end{cases}$ has the Fourier transform $F(\omega)=\sqrt{\frac{2}{\pi}}\left(\frac{\sin\omega a}{\omega}\right)$, so by the convolution theorem it follows that

$$\mathcal{F}\{(f*f)(x)\}=\sqrt{2\pi}\left[\sqrt{\frac{2}{\pi}}\left(\frac{\sin\omega a}{\omega}\right)\right]^2=2\sqrt{\frac{2}{\pi}}\left(\frac{\sin^2\omega a}{\omega^2}\right).$$

Confirm this result by calculating $(f*f)(x)$ and finding its Fourier transform.

Solution In terms of the Heaviside unit step function we can write $f(t)=H(a-|t|)$ and $f(x-t)=H(a-|x-t|)$, after which consideration of the product $f(t)f(x-t)$ shows that

$$f(t)f(x-t)=\begin{cases}1, & -a<t<x+a,\,(-2a<x<0)\\ 0, & \text{otherwise}\end{cases}$$

and

$$f(t)f(x-t)=\begin{cases}1, & x-a<t<a,\,(0<x<2a)\\ 0, & \text{otherwise.}\end{cases}$$

The required convolution is then given by

$$(f*f)(x)=\begin{cases}\int_{-a}^{x+a}dt=2a+x,\,(-2a<x<0)\\ \int_{x-a}^{a}dt=2a-x,\,(0<x<2a)\end{cases}\quad\text{and}\quad(f*f)(x)=0\text{ otherwise.}$$

Taking the Fourier transform of $(f^*f)(x)$, we have

$$\begin{aligned}\mathcal{F}\{(f*f)(x)\}&=\frac{1}{\sqrt{2\pi}}\left\{\int_{-2a}^{0}(2a+x)e^{-i\omega x}dx+\int_{0}^{2a}(2a-x)e^{-i\omega x}dx\right\}\\&=\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos 2\omega a}{\omega^2}\right),\end{aligned}$$

but $1-\cos 2\omega a=2\sin^2\omega a$, so

$$\mathcal{F}\{(f*f)(x)\}=2\sqrt{\frac{2}{\pi}}\left(\frac{\sin^2\omega a}{\omega^2}\right),$$

as required. ■

THEOREM 10.7

the Parseval relation extended to Fourier transforms

The Parseval relation for the Fourier transform If $f(x)$ has the Fourier transform $F(\omega)$, then

$$\int_{-\infty}^{\infty}|f(x)|^2dx=\int_{-\infty}^{\infty}|F(\omega)|^2d\omega.$$

Proof Setting $x=0$ in result (b) of the convolution theorem gives

$$\int_{-\infty}^{\infty}f(t)g(-t)dt=\int_{-\infty}^{\infty}F(\omega)G(\omega)d\omega.$$

As the Fourier transform is defined for both real and complex functions, it follows from the definition of the transform that $\mathcal{F}\{\bar{f}(-x)\}=\bar{F}(\omega)$, where the bar indicates

complex conjugation. If we set $g(t) = \bar{f}(-t)$, the preceding result becomes

$$\int_{-\infty}^{\infty} f(t)\bar{f}(t)dt = \int_{-\infty}^{\infty} F(\omega)\bar{F}(\omega)d(\omega),$$

or

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega,$$

and the result is proved. ■

EXAMPLE 10.9

Using the result of Example 10.3(a) and the Parseval relation, show that

$$\int_{-\infty}^{\infty} \frac{\sin^2 \omega a}{\omega^2} d\omega = \pi a.$$

Solution Substituting $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ and the corresponding Fourier transform $F(\omega) = \sqrt{\frac{2}{\pi}}\left(\frac{\sin \omega a}{\omega}\right)$ found in Example 10.3(a) into the Parseval relation gives

$$\int_{-a}^{a} 1^2 dx = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2 \omega a}{\omega^2}\right)d\omega, \text{ and so } 2a = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2 \omega a}{\omega^2}\right)d\omega \ (a > 0),$$

from which the required result follows. ■

The final theorem describes the effect on the Fourier transform of $f(x)$ caused by scaling x by a factor a, shifting x by a and shifting ω by λ.

THEOREM 10.8

some useful properties of Fourier transforms

Fourier transforms involving scaling x by a, shifting x by a, and shifting ω by λ If $f(x)$ has a Fourier transform $F(\omega)$, then

(i) $$\mathcal{F}\{f(ax)\} = \frac{1}{a}F(\omega/a) \quad (a > 0)$$

(ii) $$\mathcal{F}\{f(x-a)\} = e^{-i\omega a}F(\omega)$$

(iii) $$\mathcal{F}\{e^{i\lambda x}f(x)\} = F(\omega - \lambda)$$

Proof As the results of the theorem follow immediately from the definition of the Fourier transform, only result (i) will be proved, and the derivation of results (ii) and (iii) left as exercises. Starting from the definition of $\mathcal{F}\{f(ax)\}$ and making the variable change $u = ax$ we have

$$\mathcal{F}\{f(ax)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{-i\omega x}dx = \frac{1}{a\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-i\omega u/a}du$$
$$= \frac{1}{a}F(\omega/a)(a > 0).$$ ■

EXAMPLE 10.10

Using the function $f(x)$ and its Fourier transform $F(\omega)$ from Example 10.9, find (a) $\mathcal{F}\{f(2x)\}$, (b) $\mathcal{F}\{f(x-\pi)\}$, and (c) $\mathcal{F}\{e^{ix}f(x)\}$.

Solution Using the results of Theorem 10.8 we have:

$$\textbf{(a)}\ \mathcal{F}\{f(2x)\} = \frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\frac{\sin(\omega a/2)}{(\omega/2)}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{\sin(\omega a/2)}{\omega}\right)$$

$$\textbf{(b)}\ \mathcal{F}\{f(x-\pi)\} = e^{-i\pi\omega}\sqrt{\frac{2}{\pi}}\left(\frac{\sin\omega a}{w}\right)$$

$$\textbf{(c)}\ \mathcal{F}\{e^{ix}f(x)\} = \sqrt{\frac{2}{\pi}}\left(\frac{\sin(\omega-1)a}{\omega-1}\right)$$ ■

the Dirac delta function and the Fourier transform

The **Dirac delta function** $\delta(x)$ was introduced in connection with the Laplace transform, where it was recognized that it is not a function in the usual sense, but an *operation* that only has meaning when it appears in the integrand of a definite integral. Because of its many uses in connection with physical problems described by differential equations, we now extend its definition in a way that is suitable for use with Fourier transforms. This is accomplished by defining $\delta(x-a)$ in a symmetrical manner about $x = a$ in terms of the integrals

$$\int_{-\infty}^{\infty}\delta(x-a)f(x)dx = \int_{-\infty}^{\infty}\delta(a-x)f(x)dx = f(a), \tag{30}$$

where a is any real number.

This definition allows the Fourier transform of $\delta(x-a)$ to be represented as

$$\mathcal{F}\{\delta(x-a)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x-a)e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}}e^{-i\omega a}. \tag{31}$$

EXAMPLE 10.11

Find the Fourier transform of $f(x) = \delta(x-a)\exp[-b^2x^2]$ $(b > 0)$.

Solution By definition

$$\mathcal{F}\{\delta(x-a)\exp[-b^2x^2]\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x-a)\exp[-b^2x^2]e^{-i\omega x}dx$$

$$= \frac{1}{\sqrt{2\pi}}\exp[-(a^2b^2 + i\omega a)].$$ ■

Fourier Transforms of Partial Derivatives with Respect to *x* of a Function *f*(*x*, *t*) of Two Independent Variables

transforming partial derivatives

The Fourier transform with respect to x of a function $f(x,t)$ of two independent variables x and t, denoted by $F(\omega, t)$, is defined as

$$_x\mathcal{F}\{f(x,t)\} = F(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x,t)e^{-i\omega x}dx, \tag{32}$$

where the prefix suffix x shows the variable that is being transformed.

In (32) the variable t is not involved in the integration with respect to x, so it follows that the integral by which $f(x,t)$ is recovered from $F(\omega,t)$ and the transform of partial derivatives of $f(x,t)$ with respect to x obey the same rules as those for the function of a single variable $f(x)$. Thus, the inversion integral is given by

$$f(x,t) = {}_x\mathcal{F}^{-1}\{F(\omega,t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega,t)e^{i\omega x}\,d\omega, \tag{33}$$

and the Fourier transforms of the partial derivatives of $f(x,t)$ with respect to x are given by

$$ {}_x\mathcal{F}\left\{\frac{\partial^n}{\partial x^n}[f(x,t)]\right\} = (i\omega)^n F(\omega,t) \tag{34}$$

$$ {}_x\mathcal{F}\{x^n f(x,t)\} = i^n \frac{\partial^n}{\partial \omega^n}[F(\omega,t)] \tag{35}$$

$$ {}_x\mathcal{F}\left\{x^m\frac{\partial^n}{\partial x^n}[f(x,t)]\right\} = i^{m+n} \frac{\partial^m}{\partial \omega^m}[\omega^n F(\omega,t)]. \tag{36}$$

These results are necessary when using the Fourier transform to solve partial differential equations involving a function $f(x,t)$ of two independent variables x and t where $-\infty < x < \infty$. Once the partial differential equation has been transformed, it becomes an ordinary differential equation for $F(\omega,t)$, with t as the independent variable and ω as a parameter. When $F(\omega,t)$ has been found by solving the differential equation, the solution $f(x,t)$ of the partial differential equation is recovered from $F(\omega,t)$ by means of the inversion integral (33).

an application to the heat equation

To illustrate the application of the Fourier transform to a partial differential equation we take as an example the **one-dimensional heat equation**, the derivation of which can be found in Section 18.5. This same partial differential equation was used when developing applications of the Laplace transform in Chapter 7. The heat equation that determines the one-dimensional temperature distribution $T(x,t)$ on a plane $x =$ constant at time t in an infinite block of metal with heat conduction properties characterized by the constant κ is given by

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t}.$$

The problem we now consider is finding the temperature distribution throughout the metal at a time t when at $t = 0$ the one-dimensional temperature distribution throughout the block is given by

$$T(x,0) = f(x),$$

where $f(x)$ is a prescribed function. Our objective will be to find the temperature $T(x,t)$ on a plane $x =$ constant at a time $t > 0$ caused by the redistribution of heat as time increases.

The Laplace transform cannot be used because when applied to the spatial variable x it is only valid for $x \geq 0$, so instead we must make use of the Fourier transform with respect to x because this applies for $-\infty \leq x \leq \infty$. Taking the Fourier transform of the heat equation with respect to x gives

$$ {}_x\mathcal{F}\left\{\frac{\partial^2 T}{\partial x^2}\right\} = {}_x\mathcal{F}\frac{1}{\kappa}\left\{\frac{\partial T}{\partial t}\right\},$$

so if we apply (34) with $n = 2$, while regarding ω as a parameter, this becomes

$$-\omega^2\kappa F(\omega, t) = \frac{d}{dt}[F(\omega, t)], \quad \text{where} \quad F(\omega, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} T(x, t)e^{-i\omega x}dx.$$

The transform $F(\omega, t)$ satisfies the ordinary differential equation

$$F' + \omega^2\kappa F = 0,$$

with the solution

$$F(\omega, t) = A(\omega)\exp\{-\omega^2\kappa t\},$$

where $A(\omega)$ is to be determined (remember that ω is a constant with respect to t).

As

$$F(\omega, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} T(x, t)e^{-i\omega x}dx,$$

it follows from the initial condition that

$$F(\omega, 0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx,$$

but $F(\omega, 0) = A(\omega)$, so

$$F(\omega, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')\exp\{-i\omega x' - \omega^2\kappa t\}dx',$$

where to avoid confusion in the next step of the calculation the dummy variable x has been replaced by x'.

Applying the inversion integral to this result gives

$$\begin{aligned} T(x, t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\{i\omega x\}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')\exp\{-i\omega x' - \omega^2\kappa t\}dx'\right]d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')\left[\int_{-\infty}^{\infty} \exp\{i\omega(x - x') - \omega^2\kappa t\}d\omega\right]dx'. \end{aligned}$$

We show separately that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{i\omega(x - x') - \omega^2\kappa t\}d\omega = \sqrt{\frac{1}{4\pi\kappa t}}\exp\left\{-\frac{(x - x')^2}{4\kappa t}\right\},$$

so the required solution is seen to be given by

$$T(x, t) = \sqrt{\frac{1}{4\pi\kappa t}}\int_{-\infty}^{\infty} f(x')\exp\left\{-\frac{(x - x')^2}{4\kappa t}\right\}dx'.$$

OPTIONAL To show that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{i\omega(x - x') - \omega^2\kappa t\}d\omega = \sqrt{\frac{1}{4\pi\kappa t}}\exp\left\{-\frac{(x - x')^2}{4\kappa t}\right\}$$

we need to use a complex analysis method from Chapter 15. However, before we can use this technique, the integrand of the integral on the left must be rewritten. We multiply the exponential function by $e^P e^{-P}$ (that is, by 1), where P is to be determined later, and as a result obtain

$$\exp\{i\omega(x - x') - \omega^2\kappa t\} = e^P\exp\{-P + i\omega(x - x') - \omega^2\kappa t\}.$$

We now choose P so that the exponent in the exponential can be expressed in the form $-(\alpha - i\beta\omega)^2$. When this is done it turns out that

$$\alpha = -\frac{i(x-x')}{2\sqrt{\kappa t}}, \quad \beta = i\sqrt{\kappa t}, \quad \text{and} \quad P = -\frac{(x-x')^2}{4\kappa t},$$

so

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{i\omega(x-x') - \omega^2\kappa t\}d\omega$$
$$= \frac{1}{2\pi}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\}\int_{-\infty}^{\infty}\exp\left\{-\left(-\frac{i(x-x')}{2\sqrt{\kappa t}} + \omega\sqrt{\kappa t}\right)^2\right\}d\omega$$

Making the change of variable

$$u = -\frac{i(x-x')}{2\sqrt{\kappa t}} + \omega\sqrt{\kappa t},$$

we find that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{i\omega(x-x') - \omega^2\kappa t\}d\omega$$
$$= \frac{1}{2\pi}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\}\frac{1}{\sqrt{\kappa t}}\int_{ic-\infty}^{ic+\infty}\exp\{-u^2\}du,$$

where $c = (x-x')^2/\sqrt{(4\kappa t)}$. If we integrate $\exp\{-u^2\}$ around the rectangle with corners located at $-R$, R, $R+ic$, and $-R+ic$ in the complex plane, and proceed to the limit as $R \longrightarrow \infty$, it follows that the integrals from $-R$ to $-R+ic$ and from R to $R+ic$ vanish, so as $\exp\{-u^2\}$ has no poles inside the rectangle, we have

$$\int_{ic-\infty}^{ic+\infty}\exp\{-u^2\}du = \int_{-\infty}^{\infty}\exp\{-u^2\}du.$$

The integral on the right is related to the error function $\text{erf}(v)$ because

$$\int_0^v \exp\{-u^2\}du = \frac{\sqrt{\pi}}{2}\text{erf}(v),$$

where $\text{erf}(-v) = -\text{erf}(v)$ and $\text{erf}\,(\infty) = 1$.
Thus,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{i\omega(x-x') - \omega^2\kappa t\}d\omega$$
$$= \frac{1}{2\pi}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\}\frac{1}{\sqrt{\kappa t}}\frac{\sqrt{\pi}}{2}[\text{erf}(\infty) - \text{erf}(-\infty)]$$
$$= \frac{1}{2\pi}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\}\frac{1}{\sqrt{\kappa t}}\frac{\sqrt{\pi}}{2}2$$
$$= \sqrt{\frac{1}{4\pi\kappa t}}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\},$$

so we have shown that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{i\omega(x-x') - \omega^2\kappa t\}d\omega = \sqrt{\frac{1}{4\pi\kappa t}}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\}. \tag{37}$$

Fourier integrals are discussed in references [4.3] and [4.4]. Tables of Fourier transform pairs are given in references [4.2] and [3.11].

Summary

The Fourier transform was introduced and its most important operational properties were established. The transforms of derivatives and partial derivatives were considered, and applications were made to functions defined by an ordinary differential equation and also to the unsteady one-dimensional heat equation. Partial differential equations such as the heat equation, and the use of integral transforms in their solution, will be considered in more detail in Chapter 18.

TABLE 10.1 Fourier Transform Pairs

$f(x)$	$F(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$
1. $af(x) + bg(x)$	$aF(\omega) + bG(\omega)$
2. $f^{(n)}(x)$	$(i\omega)^n F(\omega)$
3. $x^n f(x)$	$(i)^n \dfrac{d^n}{d\omega^n}[F(\omega)]$
4. $x^m f^{(n)}(x)$	$(i)^{m+n} \dfrac{d^m}{d\omega^m}[\omega^n F(\omega)]$
5. $f(ax)(a > 0)$	$\dfrac{1}{a} F(\omega/a)$
6. $f(x - a)$	$e^{-i\omega a} F(\omega)$
7. $e^{i\lambda x} f(x)$	$F(\omega - \lambda)$
8. $(f * g)(x)$	$\sqrt{2\pi}\, F(\omega)G(\omega)$ (convolution theorem)
9. $\displaystyle\int_{-\infty}^{\infty} \lvert f(x)\rvert^2 dx$	$\displaystyle\int_{-\infty}^{\infty} \lvert F(\omega)\rvert^2 d\omega$ (Parseval relation)
10. $\begin{cases} 1, & \lvert x\rvert < a \\ 0, & \lvert x\rvert > a \end{cases}$ $(a > 0)$	$\sqrt{\dfrac{2}{\pi}}\left(\dfrac{\sin a\omega}{\omega}\right)$
11. $\dfrac{\sin ax}{x}$ $(a > 0)$	$\begin{cases} \sqrt{\dfrac{\pi}{2}}, & \lvert\omega\rvert < a \\ 0, \lvert\omega\rvert > a \end{cases}$
12. $\begin{cases} 1, & a < x < b \\ 0, & \text{otherwise} \end{cases}$ $(0 < a < b)$	$\dfrac{1}{\sqrt{2\pi}}\left(\dfrac{e^{-ia\omega} - e^{-ib\omega}}{i\omega}\right)$
13. $\begin{cases} a - \lvert x\rvert, & \lvert x\rvert < a \\ 0, & \lvert x\rvert > a \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\left(\dfrac{1 - \cos\omega a}{\omega^2}\right)$
14. $\dfrac{1}{a^2 + x^2}$ $(a > 0)$	$\sqrt{\dfrac{\pi}{2}}\,\dfrac{e^{-a\lvert\omega\rvert}}{a}$
15. $\begin{cases} e^{-ax}, & x > 0 \\ 0, & x < 0 \end{cases}$ $(a > 0)$	$\dfrac{1}{\sqrt{2\pi}}\left(\dfrac{1}{a + i\omega}\right)$
16. $\begin{cases} e^{ax}, & b < x < c \\ 0, & \text{otherwise} \end{cases}$ $(a > 0)$	$\dfrac{1}{\sqrt{2\pi}}\left[\dfrac{e^{(a-i\omega)c} - e^{(a-i\omega)b}}{a - i\omega}\right]$

(*continued*)

TABLE 10.1 (*continued*)

$f(x)$	$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$
17. $e^{-a\lvert x\rvert}$ $(a > 0)$	$\sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+\omega^2}\right)$
18. $xe^{-a\lvert x\rvert}$ $(a > 0)$	$-\sqrt{\frac{2}{\pi}}\frac{2ia\omega}{(a^2+\omega^2)^2}$
19. $\begin{cases} e^{iax}, & \lvert x\rvert < b \\ 0, & \lvert x\rvert > b \end{cases}$	$\sqrt{\frac{2}{\pi}}\left(\frac{\sin b(\omega - a)}{\omega - a}\right)$
20. $\exp(-a^2x^2)$ $(a > 0)$	$\frac{1}{a\sqrt{2}}\exp\left\{-\frac{\omega^2}{4a^2}\right\}$
21. $\begin{cases} e^{-x}x^a, & x > 0 \\ 0, & x \le 0 \end{cases}$	$\frac{\Gamma(a)}{\sqrt{2\pi}(1+i\omega)^a}$
22. $J_0(ax)$ $(a > 0)$	$\sqrt{\frac{2}{\pi}}\frac{H(a-\lvert\omega\rvert)}{(a^2-\omega^2)^{1/2}}$
23. $\delta(x-a)$ (a real)	$\frac{1}{\sqrt{2\pi}}e^{-ia\omega}$

EXERCISES 10.2

In Exercises 1 through 10 establish the Fourier transform of the stated entry in Table 10.1.

1. Entry 11.
2. Entry 12.
3. Entry 13.
4. Entry 15.
5. Entry 16.
6. Entry 17.
7. Entry 18.
8. Entry 19.
9. Entry 21.

10. Entry 22, by using the fact that $f(x) = J_0(ax)$ satisfies the Bessel's differential equation of order zero

$$xf'' + f' + a^2xf = 0 \quad (a > 0),$$

together with the standard result $\int_0^\infty J_0(ax)dx = 1/a$.

11. Use integration by parts to show that if $f(x)$ has a finite jump discontinuity at $x = a$, then $\mathcal{F}\{f'(x)\} = i\omega F(\omega) - \frac{1}{\sqrt{2\pi}}[f(a+) - f(a-)]e^{-iwa}$.

12. (a) Use the result of Exercise 11 to find the Fourier transform of $f'(x)$ given that

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(b) Calculate $f'(x)$ and use entry 12 of Table 10.1 to find $\mathcal{F}\{f'(x)\}$ directly. Hence, show that the result obtained by this direct method is in agreement with the Fourier transform found in (a). So $f'(x) = -\delta(x-1) + \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$.

10.3 Fourier Cosine and Sine Transforms

The Fourier *cosine and sine transforms* arise as special cases of the Fourier transform, according to whether $f(x)$ is even or odd. Let us start by considering the Fourier cosine transform of $f(x)$ that can be defined when $f(x)$ is an even function that is absolutely integrable over $(-\infty, \infty)$, and so possesses a Fourier transform. Result (22) of Section 10.2 can be written

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\{\cos \omega x - i \sin \omega x\}dx, \tag{38}$$

but if $f(x)$ is an even function, the product $f(x)\cos\omega x$ is also even, so its integral over $(-\infty, \infty)$ does not vanish, though the product $f(x)\sin\ \omega x$ is an odd function, so its integral over $(-\infty, \infty)$ vanishes, causing (38) to simplify to

$$F_C(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\cos\omega x dx.$$

If we use the result $f(-x) = f(x)$ to change the interval of integration to $[0, \infty)$ this last result becomes

$$F_C(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\cos\omega x dx, \tag{39}$$

Fourier sine and cosine transforms

where the integral on the right is called the **Fourier cosine transform** of $f(x)$, and to distinguish it from the ordinary Fourier transform we write $\mathcal{F}_C\{f(x)\} = F_C(\omega)$. The **Fourier cosine inversion integral** corresponding to equation (23) of Section 10.2 becomes $f(x) = \mathcal{F}_C^{-1}\{F_C(\omega)\}$, where

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_C(\omega)\cos\omega x d\omega. \tag{40}$$

inversion integrals

A similar argument applied to (16) of Section 10.2 when $f(x)$ is an odd function leads to the result

$$F_S(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\sin\omega x dx, \tag{41}$$

where the integral on the right is called the **Fourier sine transform** of $f(x)$ and we write $\mathcal{F}_S\{f(x)\} = F_S(\omega)$. The corresponding **Fourier cosine inversion integral** becomes $f(x) = \mathcal{F}_S^{-1}\{F_S(\omega)\}$, where

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_S(\omega)\sin\omega x d\omega. \tag{42}$$

The Fourier cosine transform of $f(x)$ in (39) only involves $f(x)$ for $x \geq 0$, though it was derived from the Fourier transform on the assumption that $f(x)$ was an even function defined for all x. Consequently, taking the Fourier cosine transform of an arbitrary function $f(x)$ defined for $x \geq 0$ is equivalent to transforming an *even* function $f_e(x)$ obtained from $f(x)$ by setting $f_e(x) = f(x)$ for $x \geq 0$ and defining $f_e(x)$ for $x < 0$ by $f_e(-x) = f(x)$. Similarly, the Fourier sine transform of $f(x)$ in (41) only involves $f(x)$ for $x \geq 0$, though it was derived on the assumption that $f(x)$ was an odd function. So, taking the Fourier sine transform of an arbitrary function $f(x)$ defined for $x \geq 0$ is equivalent to transforming *odd* function $f_o(x)$ obtained from $f(x)$ by setting $f_o(x) = f(x)$ for $x \geq 0$ and defining $f_e(x)$for $x < 0$ by $f_e(-x) = -f(x)$.

Because (40) and (41) have been derived from (22) of Section 10.2, it follows that whenever $f(x)$ is discontinuous, the expression on the left must be replaced by $(1/2)[f(x+0) + f(x-0)]$, because the Fourier cosine and sine transforms have the same convergence properties as the Fourier transform.

EXAMPLE 10.12

Find $\mathcal{F}_C\{e^{-ax}\}$ and $\mathcal{F}_S\{e^{-ax}\}$ when $a > 0$, and use the results with the Fourier cosine and sine inversion integrals and an interchange of variables to show that

$$\mathcal{F}_C\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}\frac{e^{-a\omega}}{a} \quad \text{and} \quad \mathcal{F}_S\left\{\frac{x}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}e^{-a\omega}.$$

Solution By definition

$$\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos\omega x dx$$

$$= \operatorname{Re}\sqrt{\frac{2}{\pi}}\left\{\int_0^\infty e^{-ax}e^{i\omega x}dx\right\} = \sqrt{\frac{2}{\pi}}\operatorname{Re}\left\{\frac{1}{a-i\omega}\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{\omega^2+a^2}\right).$$

Similarly,

$$\mathcal{F}_S\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin\omega x dx$$

$$= \operatorname{Im}\sqrt{\frac{2}{\pi}}\left\{\int_0^\infty e^{-ax}e^{i\omega x}dx\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{\omega}{\omega^2+a^2}\right).$$

Using these results in the Fourier cosine and sine inversion integrals gives

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos\omega x}{\omega^2+a^2}d\omega = \frac{2}{\pi}\int_0^\infty \frac{\omega\sin\omega x}{\omega^2+a^2}d\omega, \text{ for } a > 0,$$

so after x and ω are interchanged, these results become

$$e^{-a\omega} = \frac{2a}{\pi}\int_0^\infty \frac{\cos\omega x}{x^2+a^2}dx = \frac{2}{\pi}\int_0^\infty \frac{x\cos\omega x}{x^2+a^2}dx.$$

However,

$$\mathcal{F}_C\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos\omega x}{x^2+a^2}dx \quad \text{and} \quad \mathcal{F}_S\left\{\frac{x}{x^2+a^2}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x\sin\omega x}{x^2+a^2}dx,$$

so combining results gives

$$\mathcal{F}_C\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}\frac{e^{-a\omega}}{a} \quad \text{and} \quad \mathcal{F}_S\left\{\frac{x}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}e^{-a\omega}.$$

■

THEOREM 10.9

Linearity of the Fourier cosine and sine transforms Let the functions $f(x)$ and $g(x)$ have Fourier cosine and sine transforms, and let a and b be arbitrary constants. Then

$$\mathcal{F}_C\{af(x)+bg(x)\} = a\,\mathcal{F}_C\{f(x)\} + b\,\mathcal{F}_C\{g(x)\} = aF_C(\omega) + bG_C(\omega)$$

and

$$\mathcal{F}_S\{af(x)+bg(x)\} = a\,\mathcal{F}_S\{f(x)\} + b\,\mathcal{F}_S\{g(x)\} = aF_S(\omega) + bG_S(\omega).$$

Proof The linearity properties of the Fourier cosine and sine transforms follow directly from the linearity property of the Fourier transform from which they are derived. ■

linearity of sine and cosine transforms and the transformation of derivatives

The expressions for the Fourier cosine and sine transforms of derivatives of a function $f(x)$ are slightly more complicated than those for the Fourier transform because they involve the initial values of the function and its derivatives.

THEOREM 10.10

Fourier cosine and sine transforms of derivatives Let $f(x)$ be continuous and absolutely integrable over $[0, \infty)$ and such that $\lim_{x\to\infty} f(x) = 0$. Then if $f'(x)$ and $f''(x)$ are piecewise continuous on each finite subinterval of $[0, \infty)$,

(i) $$\mathcal{F}_C\{f'(x)\} = \omega\mathcal{F}_S\{f(x)\} - \sqrt{\frac{2}{\pi}}f(0)$$

(ii) $$\mathcal{F}_S\{f'(x)\} = -\omega\mathcal{F}_C\{f(x)\}$$

(iii) $$\mathcal{F}_C\{f''(x)\} = -\omega^2\mathcal{F}_C\{f(x)\} - \sqrt{\frac{2}{\pi}}f'(0)$$

(iv) $$\mathcal{F}_S\{f''(x)\} = -\omega^2\mathcal{F}_S\{f(x)\} + \sqrt{\frac{2}{\pi}}\omega f(0).$$

Proof The proof of each result is similar, so only result (i) will be derived in detail and outlines given for the proofs of the remaining results. To obtain (i) we integrate by parts and make use of the definition of $\mathcal{F}_C\{f(x)\}$ as follows:

$$\begin{aligned}\mathcal{F}_C\{f'(x)\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos\omega x dx\\ &= \sqrt{\frac{2}{\pi}}\left[f(x)\cos\omega x\Big|_0^\infty + \omega\int_0^\infty f(x)\sin\omega x dx\right]\\ &= -\sqrt{\frac{2}{\pi}}f(0) + \omega\mathcal{F}_S\{f(x)\}.\end{aligned}$$

Result (iii) follows from (i) by replacing f by f'. Result (ii) follows in similar fashion, and (iv) follows from (ii) by replacing f by f'. ■

When Theorem 10.10 is used in the solution of second order differential equations, the initial conditions involved will help decide whether to use the cosine or sine transform. Thus, for example, if $f(0)$ is given but $f'(0)$ is unknown, the Fourier sine transform should be used to transform $f''(x)$ because result (iv) does not involve $f'(0)$. Conversely, if $f(0)$ is unknown but $f'(0)$ is given, then the Fourier cosine transform should be used to transform $f''(x)$, because result (iii) does not involve $f(0)$.

The Fourier cosine and sine transforms have Parseval relations that are analogous to the Parseval relation for the Fourier transform given in Theorem 10.7. To arrive at the first of these results we consider two functions $f(x)$ and $g(x)$ with the respective Fourier cosine transforms $F_C(\omega)$ and $G_C(\omega)$ and, using the definition of $G_C(\omega)$, write

$$\int_0^\infty F_C(\omega)G_C(\omega)\cos\omega x d\omega = \sqrt{\frac{2}{\pi}}\int_0^\infty F_C(\omega)\cos\omega x d\omega\int_0^\infty g(x)\cos\omega x dx.$$

Changing the order of integration in the expression on the right gives

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}}\int_0^\infty F_C(\omega)\cos\omega x d\omega \int_0^\infty g(v)\cos\omega v dv \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty g(x)dx \int_0^\infty F_C(\omega)\cos\omega x\cos\omega v d\omega \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{2}[\cos\omega(x+v)+\cos\omega|x-v|]F_C(\omega)d\omega \\
&= \frac{1}{2}\int_0^\infty g(v)[f(x+v)+f(|x-v|)]dv,
\end{aligned}$$

where use has first been made of the identity $\cos u\cos v = \frac{1}{2}[\cos(u+v)+\cos(u-v)]$ and then of the Fourier cosine inversion integral.

We have established the result

$$\int_0^\infty F_C(\omega)G_C(\omega)\cos\omega x d\omega = \frac{1}{2}\int_0^\infty g(v)[f(x+v)+f(|x-v|)]dv.$$

Setting $x=0$ in this last result shows that

$$\int_0^\infty F_C(\omega)G_C(\omega)d\omega = \int_0^\infty f(v)g(v)dv. \tag{43}$$

The **Parseval relation** for the **Fourier cosine transform** follows from this result by identifying $g(v)$ with $\bar{f}(v)$, for then (43) becomes

$$\int_0^\infty |F_C(\omega)|^2 d\omega = \int_0^\infty |f(x)|^2 dx, \tag{44}$$

where in the last integral the dummy variable v has been replaced by x.

A similar argument involving the Fourier sine transform establishes the corresponding results

$$\int_0^\infty F_S(\omega)G_S(\omega)d\omega = \int_0^\infty f(v)g(v)dv \tag{45}$$

and the **Parseval relation** for the **Fourier sine transform**

$$\int_0^\infty |F_S(\omega)|^2 d\omega = \int_0^\infty |f(x)|^2 dx. \tag{46}$$

We have arrived at the following theorem.

THEOREM 10.11

the Parseval relation extended to Fourier sine and cosine transforms

The Parseval relation for the Fourier cosine and sine transforms Let $f(x)$ have the respective Fourier cosine and sine transforms $F_C(\omega)$ and $F_S(\omega)$. Then the Parseval relation for the Fourier cosine transform is

$$\int_0^\infty |F_C(\omega)|^2 d\omega = \int_0^\infty |f(x)|^2 dx,$$

and the Parseval relation for the Fourier sine transform is

$$\int_0^\infty |F_S(\omega)|^2 d\omega = \int_0^\infty |f(x)|^2 dx.$$

Results (44) and (46) often provide a simple way of evaluating improper integrals, as shown by the following example.

EXAMPLE 10.13

Apply result (43) to $f(x) = xe^{-ax}$ and $g(x) = xe^{-bx}$, where $a > 0$, $b > 0$, given that

$$\mathcal{F}_C\{f(x)\} = \sqrt{\frac{2}{\pi}}\frac{(a^2-\omega^2)}{(a^2+\omega^2)^2} \quad \text{and} \quad \mathcal{F}_C\{g(x)\} = \sqrt{\frac{2}{\pi}}\frac{(b^2-\omega^2)}{(b^2+\omega^2)^2}.$$

Solution Substituting into (43) gives

$$\frac{2}{\pi}\int_0^\infty \frac{(a^2-\omega^2)(b^2-\omega^2)}{(a^2+\omega^2)^2(b^2+\omega^2)^2}d\omega = \int_0^\infty x^2 e^{-(a+b)x}dx,$$

and after integrating the expression on the right and multiplying by $\pi/2$ we find that

$$\int_0^\infty \frac{(a^2-\omega^2)(b^2-\omega^2)}{(a^2+\omega^2)^2(b^2+\omega^2)^2}d\omega = \frac{\pi}{(a+b)^3}.$$

This integral can be evaluated by other techniques, but the preceding method is one of the simplest.

The final theorem in this section is the analogue of Theorem 10.8, and it is useful when transforming known Fourier cosine and sine transforms.

THEOREM 10.12

shifting and scaling Fourier sine and cosine transforms

Shifting ω and scaling x in Fourier cosine and sine transforms Let $f(x)$ have the respective Fourier cosine and sine transforms $F_C(\omega)$ and $F_S(\omega)$. Then

(i) $\mathcal{F}_C\{\cos(ax)f(x)\} = \frac{1}{2}\{F_C(\omega+a) + F_C(\omega-a)\}$

(ii) $\mathcal{F}_C\{\sin(ax)f(x)\} = \frac{1}{2}\{F_S(a+\omega) + F_S(a-\omega)\}$

(iii) $\mathcal{F}_S\{\cos(ax)f(x)\} = \frac{1}{2}\{F_S(\omega+a) + F_S(\omega-a)\}$

(iv) $\mathcal{F}_S\{\sin(ax)f(x)\} = \frac{1}{2}\{F_C(\omega-a) - F_C(\omega+a)\}$

(v) $\mathcal{F}_C\{f(ax)\} = \dfrac{1}{a}F_C(\omega/a) \quad (a > 0)$

(vi) $\mathcal{F}_S\{f(ax)\} = \dfrac{1}{a}F_S(\omega/a) \quad (a > 0).$

Proof (i) $\mathcal{F}_C\{\cos(ax)f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos(\omega x)\cos(ax)f(x)dx$, but

$$\cos(ax)\cos(\omega x) = \frac{1}{2}[\cos\{(a+\omega)x\} + \cos\{(a-\omega)x\}],$$

so

$$\mathcal{F}_C\{\cos(ax)f(x)\} = \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty}\cos\{(a+\omega)x\}f(x)dx + \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty}\cos\{(a-\omega)x\}f(x)dx = \frac{1}{2}\{F_C(\omega+a)+F_C(\omega-a)\}.$$

Results (ii) to (iv) follow in similar fashion, whereas results (v) and (vi) follow from the definitions of the Fourier cosine and sine transforms after making the change of variable $u = ax$. ■

EXAMPLE 10.14

Given $f(x) = e^{-ax}$ with $a > 0$, use the results of Theorem 10.12 to find (a) $\mathcal{F}_C\{\cos bx f(x)\}$ and (b) $\mathcal{F}_S\{f(bx)\}$, when $b > 0$.

Solution

(a) Using Theorem 10.12 (i) with

$$\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{\omega^2+a^2}\right),$$

gives

$$\mathcal{F}_C\{\cos bxe^{-ax}\} = \frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\frac{a}{(\omega+b)^2+a^2}\right) + \frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\frac{a}{(\omega-b)^2+a^2}\right) = \sqrt{\frac{2}{\pi}}\frac{a(\omega^2+a^2+b^2)}{[(\omega+b)^2+a^2][(\omega-b)^2+a^2]}.$$

(b) Using Theorem 10.12 (vi) with

$$\mathcal{F}_S\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{\omega}{\omega^2+a^2}\right)$$

gives

$$\mathcal{F}_S\{f(bx)\} = \mathcal{F}_S\{e^{-abx}\} = \frac{1}{b}\sqrt{\frac{2}{\pi}}\left(\frac{\omega/b}{(\omega/b)^2+a^2}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{\omega}{\omega^2+a^2b^2}\right).$$

This result is to be expected, as it follows directly from the original result when a is replaced by ab. ■

When Fourier cosine and sine transforms are used in the solution of partial differential equations, the function to be transformed is a function of more than one variable. So, for example, the operation of taking the Fourier cosine transform of $f(x, y)$ with respect to x, denoted by $F_C(\omega, y)$, is given by

$$_x\mathcal{F}_C\{f(x, y)\} = F_C(\omega, y) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x, y)\cos\omega x dx. \tag{47}$$

Similarly, the operation of taking the Fourier sine transform of $f(x, y)$ with respect to y, denoted by $F_S(x, \omega)$, is given by

$$_y\mathcal{F}_S\{f(x,\ y)\} = F_S(x,\ \omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x,\ y) \sin \omega y dy. \tag{48}$$

As a variable that has not been transformed only appears as a parameter in the transform, it follows immediately that the rules for transforming partial derivatives follow directly from the rules for transforming derivatives of functions of a single independent variable. As a result, when interpreted in terms of a function $f(x, y)$, the entries in Theorem 10.10 take the following form.

transform of partial derivatives by Fourier sine and cosine transforms

Fourier cosine and sine transforms of partial derivatives of a function f(x, y)

$$_x\mathcal{F}_C\{f'(x,\ t)\} = \omega F_S(\omega,\ t) - \sqrt{\frac{2}{\pi}} f(0,\ t) \tag{49}$$

$$_x\mathcal{F}_S\{f'(x,\ t)\} = -\omega F_C(\omega,\ t) \tag{50}$$

$$_x\mathcal{F}_C\{f''(x,\ t)\} = -\omega^2 F_S(\omega,\ t) - \sqrt{\frac{2}{\pi}} f'(0,\ t) \tag{51}$$

$$_x\mathcal{F}_S\{f''(x,\ t)\} = -\omega^2 F_S(\omega,\ t) + \sqrt{\frac{2}{\pi}} \omega f(0,\ t) \tag{52}$$

It also follows that when transforming with respect to x partial derivatives of $f(x,\ y)$ with respect to y, the function f is transformed and the partial derivative of $f(x,\ y)$ with respect to y becomes an ordinary derivative with respect to y of the transformed function. So, for example,

$$_x\mathcal{F}_C\left\{\frac{\partial^n f(x,\ y)}{\partial y^n}\right\} = \frac{d^n F_C(\omega,\ y)}{dy^n},$$

with corresponding results for mixed derivatives.

another application to the heat equation

To provide a motivation for these results we again anticipate the discussion of partial differential equations that is to follow in Chapter 18. Our objective now will be to solve the same **initial boundary value problem** for the one-dimensional **heat equation** that was solved previously by means of the Laplace transform. The one-dimensional heat equation governing the temperature $T(x,\ t)$ in a semi-infinite slab of metal at a distance x from its plane face at time t is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t}, \tag{53}$$

and as before we will seek a solution subject to the initial condition

$$T(x,\ 0) = 0 \tag{54}$$

and the boundary condition

$$T(0,\ t) = T_0, \quad t \geq 0. \tag{55}$$

The initial condition (54) says that at time $t = 0$ all the metal in the slab is at temperature $T = 0$, whereas the boundary condition (55) says that for $t > 0$ the

plane face of the slab of metal is suddenly maintained at the constant temperature $T = T_0$.

As an initial temperature is known, but $\partial T/\partial x$ is unknown, consideration of results (49) to (52) suggests that we use the Fourier sine transform because it is valid for $x \geq 0$ and it only requires knowledge of $T(0,\ t) = T_0$. Accordingly, taking the Fourier sine transform of (53) with $\mathcal{F}_S\{T(x,\ t)\} = T_S(\omega,\ t)$, we have

$$\mathcal{F}_S\left\{\frac{\partial^2 T}{\partial x^2}\right\} = \frac{1}{\kappa}\quad \mathcal{F}_S\left\{\frac{\partial T}{\partial t}\right\},$$

so using (52) and regarding ω as a parameter (it is independent of t), we obtain

$$\kappa\left(-\omega^2 T_S(\omega,\ t) + \omega T_0\sqrt{\frac{2}{\pi}}\right) = \frac{d}{dt}[T_S(\omega,\ t)].$$

Thus, $T_S(\omega, t)$ satisfies the linear differential equation

$$T_S' + \omega^2\kappa T_S = \omega\kappa T_0\sqrt{\frac{2}{\pi}}$$

with the solution

$$T_S(\omega,\ t) = \frac{T_0}{\omega}\sqrt{\frac{2}{\pi}} + A(\omega)\exp\{-\omega^2\kappa t\},$$

where the arbitrary function $A(\omega)$ enters as the integration "constant" when $T_S(\omega,\ t)$ is integrated with respect to t, during which ω behaves as a constant.

Applying the inverse Fourier sine transform to this last result gives

$$T(x,\ t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \left\{\frac{T_0}{\omega}\sqrt{\frac{2}{\pi}} + A(\omega)\exp\{-\omega^2\kappa t\}\right\}\sin\omega x d\omega.$$

To determine $A(\omega)$ we now apply the initial condition $T(x,\ 0) = 0$ to the preceding result, which then becomes

$$0 = \sqrt{\frac{2}{\pi}}\int_0^\infty \left\{\frac{T_0}{\omega}\sqrt{\frac{2}{\pi}} + A(\omega)\right\}\sin\omega x d\omega.$$

This must be true for all ω, but this is only possible if $A(\omega) = -\frac{T_0}{\omega}\sqrt{\frac{2}{\pi}}$, and so

$$T(x, t) = T_0\sqrt{\frac{2}{\pi}}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty \left(\frac{1 - \exp(-\kappa t\omega^2)}{\omega}\right)\sin\omega x d\omega\right\}.$$

The bracketed term is the inverse Fourier sine transform of $\{[1 - \exp(-\kappa\omega^2)]/\omega\}$, so if we use entry 17 in Table 10.3, the solution becomes

$$T(x, t) = T_0 \operatorname{erfc}\left\{\frac{x}{2\sqrt{\kappa t}}\right\}.$$

This is the result that was obtained in Section 7.3 (e) (ii) by means of the Laplace transform. The result agrees with physical intuition because for any fixed x we have $\lim_{t\to\infty} \operatorname{erfc}\{\frac{x}{2\sqrt{\kappa t}}\} = 1$, showing that as $t \to \infty$, so $T(x, t) \to T_0$ the constant temperature of the plane face of the metal.

Summary

The Fourier sine and cosine transforms were introduced, their inversion integrals were stated, and the main operational properties of the transforms were established. The sine and cosine transforms of ordinary and partial derivatives were derived and applications were made to the unsteady one-dimensional heat equation.

TABLE 10.2 Fourier Cosine Transform Pairs

$f(x)$	$F_C(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x dx$
1. $af(x) + bg(x)$	$aF(\omega) + bG(\omega)$
2. $\cos(ax) f(x)$	$\frac{1}{2}\{F_C(\omega + a) + F_C(\omega - a)\}$
3. $\sin(ax) f(x)$	$\frac{1}{2}\{F_S(a + \omega) + F_S(a - \omega)\}$
4. $f(ax)$	$\frac{1}{a} F_C\left(\frac{\omega}{a}\right)(a > 0)$
5. $f'(x)$	$\omega F_S(\omega) - \sqrt{\frac{2}{\pi}} f(0)$
6. $f''(x)$	$-\omega^2 F_C(\omega) - \sqrt{\frac{2}{\pi}} f'(0)$
7. $\int_0^\infty \lvert f(x)\rvert^2 dx$	$\int_0^\infty \lvert F(\omega)\rvert^2 d\omega$ (Parseval relation)
8. $\int_0^\infty f(x)g(x)dx$	$\int_0^\infty F_C(\omega)G_C(\omega)d\omega$
9. $\begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\left(\frac{\sin a\omega}{\omega}\right)$
10. $\begin{cases} 1, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\left(\frac{\sin b\omega - \sin a\omega}{\omega}\right)$
11. $x^{\alpha-1}(0 < \alpha < 1)$	$\sqrt{\frac{2}{\pi}} \frac{\Gamma(\alpha)}{\omega^\alpha} \cos \frac{\alpha\pi}{2}$
12. $\begin{cases} x, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\left(\frac{\cos b\omega + b\omega \sin b\omega - \cos a\omega - a\omega \sin a\omega}{\omega^2}\right)$
13. $e^{-ax}(a > 0)$	$\sqrt{\frac{2}{\pi}}\left(\frac{a}{\omega^2 + a^2}\right)$
14. $xe^{-ax}(a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{(a^2 - \omega^2)}{(a^2 + \omega^2)^2}$
15. $\exp\{-ax^2\} (a > 0)$	$\frac{1}{\sqrt{2a}} \exp\left\{-\frac{\omega^2}{4a}\right\}$
16. $\frac{1}{x^2 + a^2}(a > 0)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\omega}}{a}$
17. $J_0(ax)(a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{H(a - \omega)}{(a^2 - \omega^2)^{1/2}}$
18. $\frac{\sin ax}{x}(a > 0)$	$\sqrt{\frac{2}{\pi}} H(a - \omega)$

TABLE 10.3 Fourier Sine Transform Pairs

$f(x)$	$F_S(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x dx$
1. $af(x) + bg(x)$	$aF(\omega) + bG(\omega)$
2. $\cos(ax) f(x)$	$\frac{1}{2}\{F_S(\omega + a) + F_S(\omega - a)\}$
3. $\sin(ax) f(x)$	$\frac{1}{2}\{F_C(\omega - a) - F_C(\omega + a)\}$
4. $f(ax)$	$\frac{1}{a} F_S\left(\frac{\omega}{a}\right) \quad (a > 0)$
5. $f'(x)$	$-\omega F_C(\omega)$
6. $f''(x)$	$-\omega^2 F_S(\omega) + \sqrt{\frac{2}{\pi}}\omega f'(0)$
7. $\int_0^\infty \vert f(x)\vert^2 dx$	$\int_0^\infty \vert F(\omega)\vert^2 d\omega$ (Parseval relation)
8. $\int_0^\infty f(x)g(x)dx$	$\int_0^\infty F_S(\omega)G_S(\omega)d\omega$
9. $\begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\left(\frac{1 - \cos a\omega}{\omega}\right)$
10. $\begin{cases} 1, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\left(\frac{\cos a\omega - \cos b\omega}{\omega}\right)$
11. $x^{\alpha-1}$ $(0 < \alpha < 1)$	$\sqrt{\frac{2}{\pi}} \frac{\Gamma(\alpha)}{\omega^\alpha} \sin \frac{\alpha\pi}{2}$
12. e^{-ax} $(a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{\omega}{(\omega^2 + a^2)}$
13. xe^{-ax} $(a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{2a\omega}{(\omega^2 + a^2)^2}$
14. $x \exp\{-ax^2\}$ $(a > 0)$	$\frac{\omega}{(2a)^{3/2}} \exp\left\{-\frac{\omega^2}{4a}\right\}$
15. $\frac{x}{x^2 + a^2}$ $(a > 0)$	$\sqrt{\frac{\pi}{2}} e^{-a\omega}$
16. $\frac{\cos ax}{x}$ $(a > 0)$	$\sqrt{\frac{\pi}{2}} H(\omega - a)$
17. $\operatorname{erfc}\left\{\frac{x}{2a}\right\}$ $(a > 0)$	$\sqrt{\frac{2}{\pi}}\left\{\frac{1 - \exp(-a^2\omega^2)}{\omega}\right\}$

EXERCISES 10.3

In Exercises 1 through 10 establish the Fourier cosine transform of the stated entry in Table 10.2.

1. Entry 9.
2. Entry 10.
3. Entry 11.
4. Entry 12.
5. Entry 13.
6. Entry 14.
7. Entry 15.
8. Entry 16.
9. Entry 17.
10. Entry 18.

In Exercises 11 through 15 find the Fourier cosine transform of the stated function.

11. $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & \text{otherwise.} \end{cases}$

12. $f(x) = \begin{cases} \cos x, & 0 \le x \le \pi \\ 0, & \text{otherwise.} \end{cases}$

13. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$

14. $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$

15. $f(x) = \begin{cases} 1 - x^2, & 0 \le x < 1 \\ 0, & \text{otherwise.} \end{cases}$

In Exercises 16 through 23 establish the Fourier sine transform of the stated entry in Table 10.3.

16. Entry 9.

17. Entry 10.

18. Entry 11.

19. Entry 12.

20. Entry 13.

21. Entry 14.

22. Entry 15.

23. Entry 16.

In Exercises 24 through 28 find the Fourier sine transform of the stated function.

24. $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & \text{otherwise.} \end{cases}$

25. $f(x) = \begin{cases} \cos x, & 0 \le x \le \pi \\ 0, & \text{otherwise.} \end{cases}$

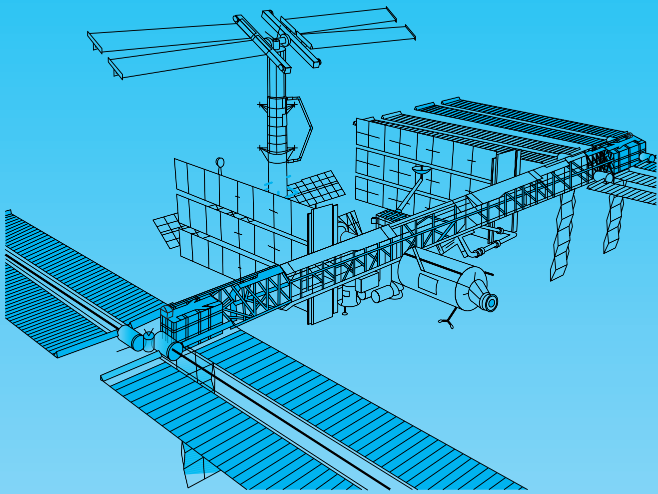
26. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$

27. $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$

28. $f(x) = \begin{cases} 1 - x^2, & 0 \le x < 1 \\ 0, & \text{otherwise.} \end{cases}$

PART FIVE

VECTOR CALCULUS



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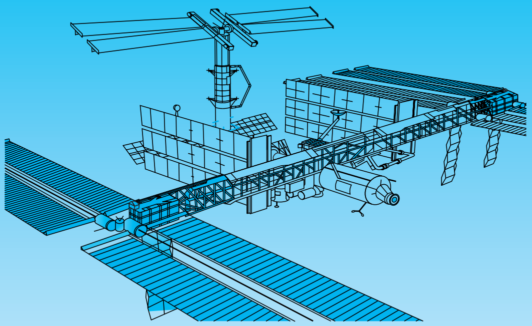
CHAPTER 11



Vector Differential Calculus

Many physical quantities that occur in engineering and science require more than a single number to characterize them. When describing quantities such as force and velocity it is necessary to specify both a magnitude and a direction, and these are examples of vector quantities, whereas the air temperature, which can be specified by giving a single number, is an example of a scalar quantity. Physical problems are often best described in terms of vectors, so the objective of this chapter is to develop the most important aspects of vector differential calculus.

Scalar and vector fields are defined in Section 11.1, and these concepts are then related to the limit, continuity, and differentiability of a vector function of a single real variable. The rules for the differentiation of vector functions of a single real variable are established and used to develop the basic geometry of space curves. The definition of the derivative at a point on a space curve is used when defining the unit tangent vector **T** to such a curve, its curvature κ, its principal normal **N**, and its binormal **B**.

The integration of scalar and vector functions of a single real variable is developed in Section 11.2, after which the line integral of a vector function of position $\mathbf{F}(x, y, z)$ is defined, and by way of example it is then used to define the circulation in a fluid flow and the flux of a vector function of position.

A directional derivative of a scalar function $w = f(x, y, z)$ is defined in Section 11.3 where its most important properties are established. The directional derivative is used when developing the concept of the gradient of f, written either grad f or ∇f, after which rules for its use are developed.

The important property of path invariance of integrals in conservative fields is proved in Section 11.4. The potential function is introduced, a test for a conservative field is given, and the determination of the related potential function is discussed, all of which concepts have important applications throughout engineering and science.

The two other vector operators divergence and curl, written div **F** and curl **F**, respectively, are defined and their physical meaning is explained in Section 11.5. The properties of the divergence operator are established, and then used to prove the properties of the most important combinations of the gradient, divergence, and curl operators.

Applications involving vector operators are often simplified if an appropriate system of coordinates is adopted. The purpose of Section 11.6 is to establish the forms taken by the gradient, divergence, and curl operators in a general system of orthogonal curvilinear coordinates, with special emphasis on cylindrical and spherical polar coordinates.

11.1 Scalar and Vector Fields, Limits, Continuity, and Differentiability

A scalar function $F(x, y, z)$ defined over some region of space D is a function that assigns to each point P_0 in D with coordinates (x_0, y_0, z_0) the *number* $F(P_0) = F(x_0, y_0, z_0)$. The set of all numbers $F(P)$ for all points P in D are said to form a **scalar field** over D. If P has position vector $\mathbf{r}$, we can write the scalar field $F(x, y, z)$ in the form $F(P) = F(\mathbf{r})$ to emphasize the fact that a *scalar* value $F(\mathbf{r})$ is associated with the position vector $\mathbf{r}$ in D. In physical problems P is usually a point in space, and in addition to depending on P, the function F often also depends on the time t, so then $F(P, t) = F(x, y, z, t)$ and in this case we can write $F(P, t) = F(\mathbf{r}, t)$. A typical example of a time dependent scalar field is provided by the temperature distribution throughout a block of metal heated in such a way that the temperatures on its sides vary with time.

scalar and vector fields

More general than a scalar field $F(x, y, z)$ is a **vector field** defined by a vector function $\mathbf{F}(x, y, z)$ over some region of space D that assigns to each point P_0 in D with coordinates (x_0, y_0, z_0) the vector $\mathbf{F}(P_0) = \mathbf{F}(x_0, y_0, z_0)$ with its tail at P_0. Functions of this type are called either **vector functions** or **vector-valued functions**, and if P has position vector $\mathbf{r}$ we can write $\mathbf{F}(P) = \mathbf{F}(\mathbf{r})$ to emphasize the fact that in this case a *vector* $\mathbf{F}(P)$ is associated with each position vector $\mathbf{r}$ in D. Like scalar fields, vector fields over D often depend on both position and the time t, so then $\mathbf{F} = \mathbf{F}(x, y, z, t)$, and in this case we can write $\mathbf{F}(P, t) = \mathbf{F}(\mathbf{r}, t)$. An example of a time dependent vector field is provided by the fluid velocity vector in the unsteady flow of water around a bridge support column, because there the velocity depends on both the position vector $\mathbf{r}$ in the water and the time t at which the velocity is observed. In general, in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, a time-dependent vector-valued function can be defined by setting

$$\mathbf{F}(\mathbf{r}, t) = f_1(\mathbf{r}, t)\mathbf{i} + f_2(\mathbf{r}, t)\mathbf{j} + f_3(\mathbf{r}, t)\mathbf{k}, \tag{1}$$

where the scalars $f_1(\mathbf{r}, t)$, $f_2(\mathbf{r}, t)$, and $f_3(\mathbf{r}, t)$ are the components of $\mathbf{F}(\mathbf{r}, t)$ that depend on both position and time and, at a point $\mathbf{r}_0$, translating the vector $\mathbf{F}(\mathbf{r}_0, t)$ until its tail is located at $\mathbf{r}_0$.

EXAMPLE 11.1

(a) The scalar function of position $F(x, y, z) = xyz^2$ for (x, y, z) inside the unit sphere $x^2 + y^2 + z^2 = 1$ defines a scalar field throughout the unit sphere.

(b) The vector-valued function $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (xyz - 2)\mathbf{k}$, for (x, y, z) inside the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, defines a vector field throughout the ellipsoid. ■

In order to perform calculus on vectors it is necessary to introduce the idea of a vector as a function. The simplest example of this kind is a vector $\mathbf{F}(t)$ of a single real variable t, which in terms of cartesian coordinates can be written

$$\mathbf{F}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}, \tag{2}$$

where the components $f_1(t)$, $f_2(t)$, and $f_3(t)$ of $\mathbf{F}(t)$ are functions of t defined over some interval $a \leq t \leq b$. Vectors of this type are called **vector functions of a single real variable**.

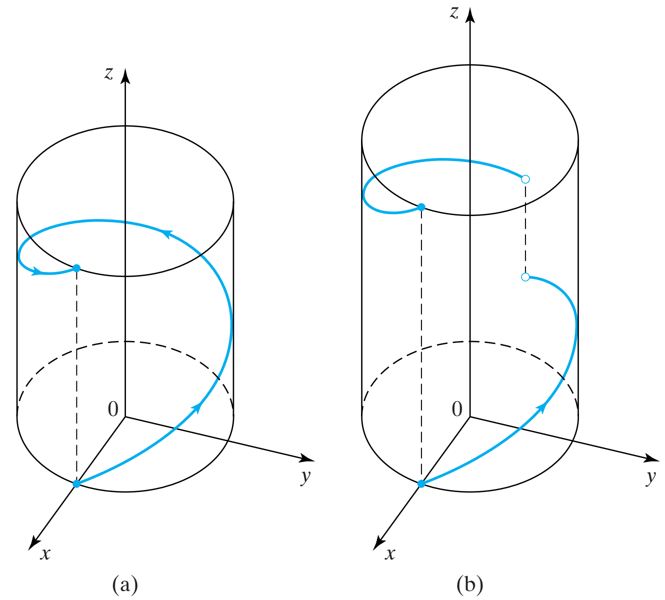


FIGURE 11.1 (a) A single turn of a helix. (b) A single turn of a broken helix.

If $\mathbf{F}(t)$ is regarded as a position vector $\mathbf{r}(t)$ in space, (2) can be interpreted as a curve in space traced out by the tip of the vector $\mathbf{r}(t)$ as t increases from a to b. Notice that a *sense* (of direction) along the curve is determined by the direction in which $\mathbf{r}(t)$ moves along the curve as t increases. When the components of $\mathbf{r}(t)$ are all continuous functions the curve, or path, traced out by the tip of $\mathbf{r}(t)$ will be an unbroken curve in space and $\mathbf{r}'(t) \neq \mathbf{0}$, though the curve will only be *smooth* if in addition to the components of $\mathbf{r}(t)$ being continuous they are also continuously differentiable for $a \leq t \leq b$, but more will be said about this later. If t is allowed to *decrease* from b to a, then the sense along the curve is *reversed*, and this fact will be important later when line integrals are considered.

EXAMPLE 11.2

(a) When interpreted as a position vector, the vector function of a single real variable $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ for $0 \leq t \leq 2\pi$ describes a single turn of the space curve called a **helix** that is shown in Fig. 11.1(a). The fact that each component of $\mathbf{r}(t)$ is both continuous and continuously differentiable and $|d\mathbf{r}/dt| \neq 0$ ensures that the helix is a smooth curve. The form of the helix can be visualized by recognizing that, as t increases, so the projection of $\mathbf{r}(t)$ onto the (x, y)-plane given by the vector $\mathbf{r}_{(x,y)}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$ moves once in a counterclockwise direction around a unit circle centered on the origin, while the $\mathbf{k}$ component increases linearly with t.

(b) The vector function of a single real variable $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \theta(t + H(t - \pi))\mathbf{k}$ for $0 \leq t \leq 2\pi$, where $H(t)$ is the Heaviside unit step function, has a discontinuous $\mathbf{k}$ component, and so describes the broken helix shown in Fig. 11.1(b), where the jump in the $\mathbf{k}$ component of $\mathbf{r}(t)$ occurs at $t = \pi$. ■

It is important to recognize that because vector quantities are independent of a coordinate system, vector-valued functions and vector fields do not depend for their existence on any particular coordinate system. The choice of coordinate

system used to describe vector functions is usually taken to be the one that is most appropriate for the geometry of the situation involved. So, for example, when a vector of interest depends only on distance along a straight axis and on the position on a circle centered on the axis and lying in a plane normal to the axis, it is natural to describe it in terms of the cylindrical polar coordinates (r, θ, z).

To make further progress it is necessary to generalize the related concepts of the limit and continuity of a real function of a single real variable to vector functions of a single real variable.

Limits and continuity of vector functions of a single real variable

limits and continuity of vector functions

A vector function of a single real variable $\mathbf{F}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ is said to have $\mathbf{L}$ as its **limit** at t_0, written $\lim_{t\to t_0} \mathbf{F}(t) = \mathbf{L}$, where $\mathbf{L} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k}$, if

$$\lim_{t\to t_0} f_1(t) = L_1, \quad \lim_{t\to t_0} f_2(t) = L_2, \quad \text{and} \quad \lim_{t\to t_0} f_3(t) = L_3.$$

If, in addition, the vector function is defined at t_0 and $\lim_{t\to t_0} \mathbf{F}(t) = \mathbf{F}(t_0)$, then $\mathbf{F}(t)$ is said to be **continuous** at t_0. A vector function $\mathbf{F}(t)$ that is continuous for each t in the interval $a \le t \le b$ is said to be continuous over the interval. A vector function of a single real variable that is not continuous at a point t_0 is said to be **discontinuous** at t_0.

It can be seen from the preceding definitions that the limit and continuity properties of a parametrically defined vector function can be determined by examination of the behavior of its components. So, for example, the parametrically defined vector function describing the helix in Example 11.1(a) is seen to be continuous, whereas the broken helix in Example 11.1(b) is seen to be discontinuous at one point because of the behavior of its $\mathbf{k}$ component when $t = \pi$.

The notion of a limit of a vector function of a single real variable leads naturally to the definition of the differentiability of such a function. Returning to (2) we see that if t is increased to $t + \Delta t$, the change $\Delta\mathbf{F}$ produced in $\mathbf{F}$ is

$$\begin{aligned}\Delta\mathbf{F} &= \mathbf{F}(t+\Delta t) - \mathbf{F}(t)\\ &= \{f_1(t+\Delta t)\mathbf{i} + f_2(t+\Delta t)\mathbf{j} + f_3(t+\Delta t)\mathbf{k}\} - \{f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}\},\end{aligned}$$

so

$$\frac{\Delta\mathbf{F}}{\Delta t} = \left(\frac{f_1(t+\Delta t) - f_1(t)}{\Delta t}\right)\mathbf{i} + \left(\frac{f_2(t+\Delta t) - f_2(t)}{\Delta t}\right)\mathbf{j} + \left(\frac{f_3(t+\Delta t) - f_3(t)}{\Delta t}\right)\mathbf{k}.$$

If the functions $f_1(t)$, $f_2(t)$, and $f_3(t)$ are differentiable, by letting $\Delta t \to 0$ it follows at once that the derivative of $\mathbf{F}(t)$, denoted by $d\mathbf{F}/dt$, can be expressed in terms of the derivatives of the components of $\mathbf{F}(t)$ as

$$\frac{d\mathbf{F}}{dt} = \frac{df_1}{dt}\mathbf{i} + \frac{df_2}{dt}\mathbf{j} + \frac{df_3}{dt}\mathbf{k}. \tag{3}$$

We have arrived at the following definitions of the differentiability of $\mathbf{F}(t)$ and the derivative $d\mathbf{F}/dt$.

Differentiability and the derivative of a vector function of a single real variable

The vector function of a single real variable $\mathbf{F}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ defined over the interval $a \le t \le b$ is said to be **differentiable** at a point t_0 in the interval if its components are differentiable at t_0. It is said to be **differentiable over the interval** if it is differentiable at each point of the interval, and when $\mathbf{F}(t)$ is differentiable its **derivative** with respect to t is

$$\frac{d\mathbf{F}}{dt} = \frac{df_1}{dt}\mathbf{i} + \frac{df_2}{dt}\mathbf{j} + \frac{df_3}{dt}\mathbf{k}.$$

If $\mathbf{F}(t)$ is continuous over $a \le t \le b$, but $d\mathbf{F}/dt$ is discontinuous at a point t_0 in the interval, the derivative $d\mathbf{F}/dt$ will only be defined in the one-sided sense to the left and right of t_0 at the points $t = t_0 - 0$ and $t = t_0 + 0$.

When $d\mathbf{F}/dt$ is differentiable, the second order derivative $d^2\mathbf{F}/dt^2$ is defined as

$$\frac{d^2\mathbf{F}}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{F}}{dt}\right)$$

and, in general, provided the derivatives exist,

$$\frac{d^n\mathbf{F}}{dt^n} = \frac{d}{dt}\left(\frac{d^{n-1}\mathbf{F}}{dt^{n-1}}\right), \quad \text{for } n \ge 2.$$

If $\mathbf{F}(t)$ is taken to be a differentiable position vector $\mathbf{r}(t)$, it follows from the definition of a derivative that $d\mathbf{r}/dt$ is a vector that is tangent to the point $\mathbf{r}(t)$ on the curve Γ traced out by the tip of the vector as t increases from $t = a$ to $t = b$. This situation, illustrated in Fig. 11.2, shows the relationship between $\mathbf{r}(t + \Delta t)$, $\mathbf{r}(t)$, and $\Delta\mathbf{r}$ before proceeding to the limit as $\Delta t \to 0$. It can be seen from this that as $\Delta t \to 0$, so $\Delta\mathbf{r}$ tends to coincidence with the tangent line T to the curve Γ at the point $\mathbf{r}(t)$. Furthermore, if $\mathbf{r}(t)$ is a position vector in space and t is the time, $d\mathbf{r}/dt$ is the **velocity** of the point with position vector $\mathbf{r}(t)$ and $d^2\mathbf{r}/dt^2$ is its **acceleration**.

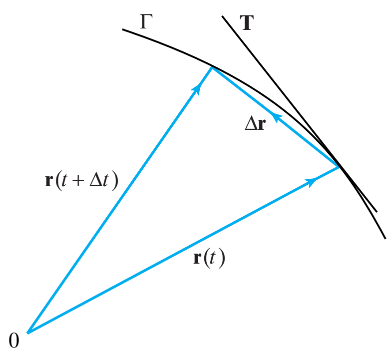


FIGURE 11.2 As $\Delta t \to 0$, so the vector $\Delta\mathbf{r}$ tends to coincidence with the tangent line T to the space curve Γ at $\mathbf{r}(t)$.

The differentiability properties of vector functions of a single real variable have been seen to be determined by the differentiability properties of the components. Consequently, as $\mathbf{F}(t)$ is a linear combination of its components in the $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ directions, it follows that the rules for the differentiation of vector functions of a single real variable follow directly by applying the rules for the differentiation of a real function of a single real variable to each component in turn. The theorem that follows summarizes the basic rules for differentiation, and because vectors are independent of a coordinate system the results can be formulated without reference to a coordinate system.

THEOREM 11.1

differentiation of vector functions

Differentiation of vector functions of a single real variable Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be differentiable functions of t over some interval $a \le t \le b$, with $\mathbf{C}$ an arbitrary constant vector and c an arbitrary constant scalar. Then rules for differentiation of vector functions of a single real variable over the interval $a \le t \le b$ are:

(i) $$\frac{d\mathbf{C}}{dt} = \mathbf{0}$$ (differentiation of a constant vector)

(ii) $$\frac{d}{dt}(c\mathbf{u}) = c\frac{d\mathbf{u}}{dt}$$ (differentiation of a vector scaled by c)

(iii) $$\frac{d}{dt}(\mathbf{u} \pm \mathbf{v}) = \frac{d\mathbf{u}}{dt} \pm \frac{d\mathbf{v}}{dt}$$ (differentiation of a sum or difference)

(iv) $$\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) = \frac{d\mathbf{u}}{dt} \cdot \mathbf{v} + \mathbf{u} \cdot \frac{d\mathbf{v}}{dt}$$ (differentiation of a dot product)

(v) $$\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \frac{d\mathbf{u}}{dt} \times \mathbf{v} + \mathbf{u} \times \frac{d\mathbf{v}}{dt}$$ (differentiation of a cross product)

(vi) If $\mathbf{u}(t)$ is a differentiable function of t and $t = t(s)$ is a differentiable function of s, then

$$\frac{d\mathbf{u}}{ds} = \frac{d\mathbf{u}}{dt}\frac{dt}{ds}$$

or, explicitly, if $\mathbf{u}(t) = u_1(t)\mathbf{i} + u_2(t)\mathbf{j} + u_3(t)\mathbf{k}$, then

$$\frac{d\mathbf{u}}{ds} = \frac{du_1}{dt}\frac{dt}{ds}\mathbf{i} + \frac{du_2}{dt}\frac{dt}{ds}\mathbf{j} + \frac{du_3}{dt}\frac{dt}{ds}\mathbf{k}$$

(the chain rule for differentiation of $\mathbf{u}(t)$).

Proof The proof of each result is straightforward and similar, so only the proof of result (iv) will be given, and for convenience the vectors $\mathbf{u}$ and $\mathbf{v}$ will be expressed in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. The proofs of the remaining results will be left as exercises.

Letting $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, we have

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

We now differentiate the scalar function $\mathbf{u} \cdot \mathbf{v}$ with respect to t, using the result

$$\frac{d(u_iv_i)}{dt} = \frac{du_i}{dt}v_i + u_i\frac{dv_i}{dt}, \quad \text{for } i = 1, 2, 3,$$

which when $i = 1$ can be written

$$\frac{d(u_1v_1\mathbf{i})}{dt} = \left(\frac{du_1}{dt}\mathbf{i}\right)\cdot(v_1\mathbf{i}) + (u_1\mathbf{i})\cdot\left(\frac{dv_i}{dt}\mathbf{i}\right),$$

with corresponding results for $d(u_2v_2)/dt$ and $d(u_3v_3)/dt$. Summing the results for $d(u_iv_i)dt$ corresponding to $i = 1, 2, 3$, we arrive at result (iv), and the proof is complete. ■

EXAMPLE 11.3

Given that $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, find the first three derivatives of $\mathbf{r}$ with respect to t.

Solution $\frac{d\mathbf{r}}{dt} = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$, $\frac{d^2\mathbf{r}}{dt^2} = -\cos t\mathbf{i} - \sin t\mathbf{j}$, and $\frac{d^3\mathbf{r}}{dt^3} = \sin t\mathbf{i} - \cos t\mathbf{j}$. ■

EXAMPLE 11.4

Given that $\mathbf{u} = t\mathbf{i} - 2t\mathbf{j} + t^2\mathbf{k}$, $\mathbf{v} = t\mathbf{j} + 3t\mathbf{k}$ and $\mathbf{w} = t\mathbf{i} - t^2\mathbf{k}$, find

$$\frac{d}{dt}[(\mathbf{u}\cdot\mathbf{v})\mathbf{w}].$$

Solution The scalar $\mathbf{u}\cdot\mathbf{v} = -2t^2 + 3t^3$, so $(\mathbf{u}\cdot\mathbf{v})\mathbf{w} = (3t^4 - 2t^3)\mathbf{i} - (3t^5 - 2t^4)\mathbf{k}$, and so

$$\frac{d}{dt}[(\mathbf{u}\cdot\mathbf{v})\mathbf{w}] = (12t^3 - 6t^2)\mathbf{i} - (15t^4 - 8t^3)\mathbf{k}.$$ ■

vector differential

The concept of a **vector differential** is often useful, and by analogy with the real variable calculus, if $\mathbf{F}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$, the vector differential $d\mathbf{F}$ is defined as

$$d\mathbf{F} = \left(\frac{df_1}{dt}\mathbf{i} + \frac{df_2}{dt}\mathbf{j} + \frac{df_3}{dt}\mathbf{k}\right)dt. \tag{4}$$

A simple and useful application of the vector differential is to the element of arc length along a space curve Γ defined by the position vector $\mathbf{r}(t) = x_1(t)\mathbf{i} + x_2(t)\mathbf{j} + x_3(t)\mathbf{k}$ for $t \geq t_0$. If s is the arc length measured along Γ from some fixed point, then by applying Pythagoras' theorem to the differential elements

$$dx_1 = \frac{dx_1}{dt}dt, \quad dx_2 = \frac{dx_2}{dt}dt, \quad \text{and} \quad dx_3 = \frac{dx_3}{dt}dt,$$

it is seen from Fig. 11.3 that the differential element of arc length ds along Γ is given by

$$ds = \left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2\right]^{1/2} dt, \tag{5}$$

and so

$$\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right| = \left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2\right]^{1/2}. \tag{6}$$

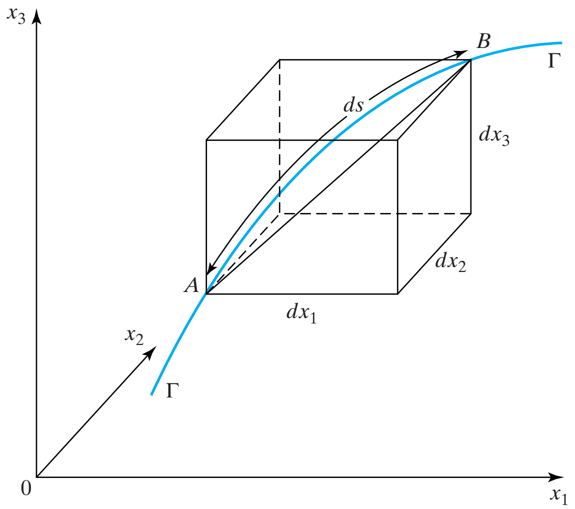


FIGURE 11.3 The geometrical relationship between the differentials ds, dx_1, dx_2, and dx_3.

This result shows that when t is the time and $\mathbf{r}(t)$ is a position vector in space, $\frac{ds}{dt} = |\frac{d\mathbf{r}}{dt}|$ is the *speed* with which the tip of position vector $\mathbf{r}(t)$ traces out a space curve Γ.

tangent vector

Examination of Fig. 11.2 and consideration of the definition of $d\mathbf{r}/dt$ shows that the *unit tangent vector* $\mathbf{T}$ along Γ as a function of t is given by

$$\boxed{\mathbf{T} = \frac{d\mathbf{r}}{dt}\bigg/\left|\frac{d\mathbf{r}}{dt}\right|,} \tag{7}$$

and as $ds/dt = |d\mathbf{r}/dt|$, this can be rewritten in the form

$$\frac{d\mathbf{r}}{dt} = \frac{ds}{dt}\mathbf{T}. \tag{8}$$

EXAMPLE 11.5

If $\mathbf{r}(t)$ is a position vector and t is the time, find the velocity, speed, and acceleration of a particle with position vector $\mathbf{r}(t) = a\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j}$, where a and ω are constants, and interpret the results.

Solution We have $|\mathbf{r}(t)| = (a^2\cos^2\omega t + a^2\sin^2\omega t)^{1/2} = a$, so as the motion is two-dimensional in the plane containing $\mathbf{i}$ and $\mathbf{j}$, it takes place in a circle of radius a with its center at the origin of the coordinate system. Differentiation of $\mathbf{r}(t)$ gives

$$\frac{d\mathbf{r}}{dt} = -\omega a\sin\omega t\mathbf{i} + \omega a\cos\omega t\mathbf{j} \quad \text{and} \quad \frac{d^2\mathbf{r}}{dt^2} = -\omega^2 a\cos\omega t\mathbf{i} - \omega^2 a\sin\omega t\mathbf{j}.$$

The speed $ds/dt = |d\mathbf{r}/dt| = \omega a$ is constant, and the velocity $d\mathbf{r}/dt$ is seen to be tangential to the circular path, because $\mathbf{r}\cdot(d\mathbf{r}/dt) = 0$. The acceleration $d^2\mathbf{r}/dt^2$ is proportional to $\mathbf{r}$, but oppositely directed, so it is always directed toward the origin. Figure 11.4 illustrates the relationship between the velocity and acceleration as the particle moves around the circle at a constant speed ωa.

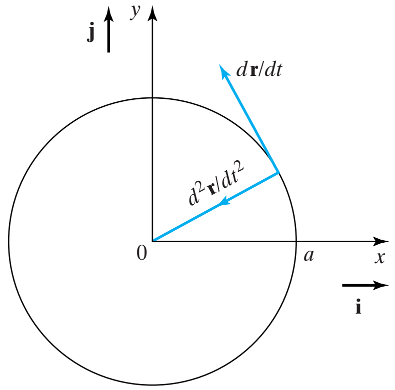


FIGURE 11.4 Uniform motion around the circle $\mathbf{r} = a\cos\omega t\,\mathbf{i} + a\sin\omega t\,\mathbf{j}$.

In dealing with the geometry of a space curve Γ, it is often convenient to specify the position vector $\mathbf{r}$ of a point on the curve in terms of the arc length s measured along the curve from some fixed point, so that then $\mathbf{r} = \mathbf{r}(s)$. When $\mathbf{r}$ is expressed in this manner the equation $\mathbf{r} = \mathbf{r}(s)$ is called the **intrinsic equation** of Γ. In addition to the unit tangent $\mathbf{T}$ at any point $\mathbf{r} = \mathbf{r}(s)$ of Γ, two other important unit vectors $\mathbf{N}$ and $\mathbf{B}$ can also be defined at that point.

intrinsic vector equation

To arrive at definitions of vectors $\mathbf{N}$ and $\mathbf{B}$, we start from the fact that as $\mathbf{T}$ is a unit vector $\mathbf{T}\cdot\mathbf{T} = 1$, so differentiating with respect to t and using Theorem 11.1(iv) we have

$$\frac{d\mathbf{T}}{ds}\cdot\mathbf{T} + \mathbf{T}\cdot\frac{d\mathbf{T}}{ds} = 0.$$

However, as the scalar product is commutative, this last result is seen to be equivalent to

$$\mathbf{T}\cdot\frac{d\mathbf{T}}{ds} = 0,$$

showing that $\mathbf{T}$ and $d\mathbf{T}/ds$ are orthogonal. The unit vector $\mathbf{N}$ in the direction of $d\mathbf{T}/ds$ at a point $\mathbf{r} = \mathbf{r}(s)$ on Γ is called the **principal normal** to Γ at $\mathbf{r}(s)$, and so

$$\mathbf{N} = \frac{d\mathbf{T}}{ds}\Bigg/\left|\frac{d\mathbf{T}}{ds}\right| \quad \text{for} \quad \left|\frac{d\mathbf{T}}{ds}\right| \neq 0. \tag{9}$$

When the connection between $d\mathbf{T}/ds$ and $\mathbf{N}$ at a point $\mathbf{r} = \mathbf{r}(s)$ on Γ is written in the form

$$\frac{d\mathbf{T}}{ds} = \kappa(s)\mathbf{N}, \tag{10}$$

curvature, normal and binormal

the nonnegative number $\kappa(s)$ is called the **curvature** of the curve Γ at $\mathbf{r} = \mathbf{r}(s)$, and $\rho(s) = 1/\kappa(s)$ is called the **radius of curvature** of the curve Γ at $\mathbf{r} = \mathbf{r}(s)$. As $\mathbf{N}$ is a

unit vector, taking the modulus of (10) gives

$$\kappa(s) = \left|\frac{d\mathbf{T}}{ds}\right|. \tag{11}$$

In the case of a smooth plane curve Γ, the circle of curvature at a point P on Γ is tangent to Γ at P with radius $\rho = 1/\kappa$, and such that its center lies on the concave side of Γ.

If the curvature is required in terms of the parameter t, the relationship between $\kappa(s)$ and $\kappa(t)$ follows from the chain rule

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt},$$

showing that

$$\left|\frac{d\mathbf{T}}{dt}\right| = \kappa(t)\left|\frac{ds}{dt}\right|. \tag{12}$$

As $dt/ds = 1/(ds/dt) = 1/|d\mathbf{r}/dt|$, this last result can be written in the convenient form

$$\kappa(t) = \left|\frac{d\mathbf{T}}{dt}\right| \Big/ \left|\frac{d\mathbf{r}}{dt}\right|. \tag{13}$$

Finally, the vector $\mathbf{B}$, defined as

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}, \tag{14}$$

is called the **unit binormal** to the curve Γ at $\mathbf{r} = \mathbf{r}(s)$. The three unit vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at a point $\mathbf{r} = \mathbf{r}(s)$ on the space curve Γ form a *triad* of mutually orthogonal unit vectors whose orientation depends on the location of the point on Γ. When studying the geometry of space curves it proves to be more convenient to use the unit vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$, whose orientation depends on the point on the curve under consideration, than a fixed reference system of unit vectors such as $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. ■

EXAMPLE 11.6 Show that the straight line $\mathbf{r}(t) = at\mathbf{i} + bt\mathbf{j} + ct\mathbf{k} + \mathbf{C}$, with a, b, and c scalar constants and $\mathbf{C}$ a constant vector, has an infinite radius of curvature at every point.

Solution Differentiation shows that $|d\mathbf{r}/dt| = (a^2 + b^2 + c^2)^{1/2} \neq 0$, and the tangent vector $\mathbf{T} = d\mathbf{r}/dt/|d\mathbf{r}/dt| = (a\mathbf{i} + b\mathbf{j} + c\mathbf{k})/(a^2 + b^2 + c^2)^{1/2}$, so $d\mathbf{T}/dt \equiv 0$, and $\mathbf{N}$ has to be chosen arbitrarily except for $\mathbf{T}\cdot\mathbf{N} = 0$. Consequently, from (13) $\kappa(t) \equiv 0$, and so the radius of curvature $\rho(t) = 1/\kappa(t) = \infty$ for all t. ■

EXAMPLE 11.7 Find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, and $\kappa(t)$ for the helix $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\mathbf{k}$.

Solution From $ds/dt = |d\mathbf{r}/dt|$ we have

$$ds/dt = [(-a\sin t)^2 + (a\cos t)^2 + b^2]^{1/2} = (a^2 + b^2)^{1/2},$$

and so

$$\mathbf{T} = \frac{d\mathbf{r}}{dt}\bigg/\frac{ds}{dt} = \frac{1}{(a^2+b^2)^{1/2}}(-a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k}).$$

By definition,

$$\mathbf{N} = \frac{d\mathbf{T}}{ds}\bigg/\left|\frac{d\mathbf{T}}{ds}\right| = \frac{d\mathbf{T}}{dt}\frac{dt}{ds}\bigg/\left|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}\right| = \frac{d\mathbf{T}}{dt}\bigg/\left|\frac{d\mathbf{T}}{dt}\right| = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

and

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \frac{1}{(a^2+b^2)^{1/2}}(b\sin t\mathbf{i} - b\cos t\mathbf{j} + a\mathbf{k}).$$

A simple calculation shows that $|d\mathbf{T}/dt| = a/(a^2+b^2)^{1/2}$, $|d\mathbf{r}/dt| = (a^2+b^2)^{1/2}$, so it follows from (13) that the curvature $\kappa(t) = a/(a^2+b^2)$ for all t. This is to be expected, because the uniform shape of the helix implies that the curvature, and hence the radius of curvature, are constant along the helix. ■

Summary

Scalar and vector fields have been introduced, vector functions of a single real variable have been defined, and their differentiability properties have been derived. Applications to dynamics and the geometry of space curves have been made.

EXERCISES 11.1

In Exercises 1 through 6 find the first and second derivatives of the function and their values at the given value of t.

1. $\mathbf{r} = t\sin t\mathbf{i} + t\cos t\mathbf{j} + t^2\mathbf{k}, \quad t = \pi/2.$
2. $\mathbf{r} = (1+t^2)\mathbf{i} + e^{-2t}\mathbf{j} + \sqrt{t}\mathbf{k}, \quad t = 1.$
3. $\mathbf{r} = (2-\cos^2 t)\mathbf{i} + \sin^2 t\mathbf{j} + (\pi - t)\mathbf{k}, \quad t = \pi/4.$
4. $\mathbf{r} = \ln(1+t)\mathbf{i} + \ln(1+r^2)\mathbf{j} + e^{3t}\mathbf{k}, \quad t = 0.$
5. $\mathbf{r} = (t-\sin t)\mathbf{i} + (1-\cos t)\mathbf{j}, \quad t = \pi/2$ (a **cycloid**).
   Notice that $\mathbf{r}$ is arbitrarily many times differentiable, yet the cycloid has cusps for $t = n\pi$.
6. $\mathbf{r} = 4\cos t\mathbf{i} + 3\sin t\mathbf{j} + 2t\mathbf{k}, \quad t = \pi/4$ (an **elliptical "helix"**).
7. Prove result (iii) in Theorem 11.1 by expressing the vectors in terms of their cartesian components.
8. Prove result (v) in Theorem 11.1 by expressing the vectors in terms of their cartesian components.
9. Given that $\mathbf{r} = t\mathbf{i} + 3t^2\mathbf{j} - (t-1)\mathbf{k}$ and $t = \ln(1+s^2)$, use result (vi) in Theorem 11.1 to find $d\mathbf{r}/ds$.
10. Given that $\mathbf{r} = \sin t\mathbf{i} + \cos t\mathbf{j} + \tan t\mathbf{k}$ and $t = 2+s^2$, use result (vi) in Theorem 11.1 to find $d\mathbf{r}/ds$.
11. A particle has a position vector at time t given by

$$\mathbf{r} = t^2\mathbf{i} + 4\cos 2t\mathbf{j} + 3\sin 2t\mathbf{k}.$$

   Find the component of its velocity in the direction $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ at time t.
12. A particle has a position vector at time t given by

$$\mathbf{r} = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + (t^2 - 2)\mathbf{k}.$$

   Find the component of its velocity in the direction $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ at time t.
13. If $\phi(t)$ is a differentiable function of t and $\mathbf{u}(t)$ is a differentiable parametrically defined function of t, prove that

$$\frac{d}{dt}(\phi\mathbf{u}) = \phi\frac{d\mathbf{u}}{dt} + \frac{d\phi}{dt}\mathbf{u}.$$

14. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are differentiable parametrically defined functions of t, prove that

$$\frac{d}{dt}(\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})) = \mathbf{u}\cdot\left(\mathbf{v}\times\frac{d\mathbf{w}}{dt}\right) + \mathbf{u}\cdot\left(\frac{d\mathbf{v}}{dt}\times\mathbf{w}\right) + \frac{d\mathbf{u}}{dt}\cdot(\mathbf{v}\times\mathbf{w}),$$

   where the order in the products must be preserved.
15. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are differentiable parametrically defined functions of t, prove that

$$\frac{d}{dt}(\mathbf{u}\times(\mathbf{v}\times\mathbf{w})) = \mathbf{u}\times\left(\mathbf{v}\times\frac{d\mathbf{w}}{dt}\right) + \mathbf{u}\times\left(\frac{d\mathbf{v}}{dt}\times\mathbf{w}\right) + \frac{d\mathbf{u}}{dt}\times(\mathbf{v}\times\mathbf{w}),$$

   where the order in the products must be preserved.

16. If $\mathbf{u}$ is a differentiable parametrically defined function of t, prove that

$$\frac{d\mathbf{u}}{dt}\times\frac{d}{dt}\left(\frac{d\mathbf{u}}{dt}\times\frac{d^2\mathbf{u}}{dt^2}\right)=\frac{d\mathbf{u}}{dt}\left(\frac{d\mathbf{u}}{dt}\cdot\frac{d^3\mathbf{u}}{dt^3}\right)-\frac{d^3\mathbf{u}}{dt^3}\left(\frac{d\mathbf{u}}{dt}\right)^2.$$

17. If $\mathbf{u}$ is a differentiable parametrically defined function of t, prove that

$$\frac{d\mathbf{u}}{dt}\times\frac{d}{dt}\left(\mathbf{u}\times\frac{d\mathbf{u}}{dt}\right)=\mathbf{u}\left(\frac{d\mathbf{u}}{dt}\cdot\frac{d^2\mathbf{u}}{dt^2}\right)-\frac{d^2\mathbf{u}}{dt^2}\left(\mathbf{u}\cdot\frac{d\mathbf{u}}{dt}\right).$$

18. Given that $\phi(t)=t^2\cos t$ and $\mathbf{u}=\sin t\mathbf{i}+2\cos t\mathbf{j}+(1+t^2)^{1/2}\mathbf{k}$, use the result of Exercise 13 to find $\frac{d}{dt}(\phi\mathbf{u})$, and confirm the result by direct differentiation of $\phi\mathbf{u}$ with respect to t.

19. Given that $\mathbf{u}=2t\mathbf{i}-t^2\mathbf{j}+\mathbf{k}$, $\mathbf{v}=2\mathbf{i}+3t\mathbf{j}+t\mathbf{k}$, and $\mathbf{w}=t\mathbf{i}+2t\mathbf{j}-t\mathbf{k}$, use the result of Exercise 14 to find $\frac{d}{dt}(\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}))$. Confirm the result by finding $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ and differentiating the result with respect to t.

20. Given that $\mathbf{u}=t\mathbf{i}-t\mathbf{j}+t^2\mathbf{k}$, $\mathbf{v}=-t\mathbf{i}+2t\mathbf{j}-t^2\mathbf{k}$, and $\mathbf{w}=2t\mathbf{i}-2t\mathbf{j}+t\mathbf{k}$, use the result of Exercise 15 to find $\frac{d}{dt}(\mathbf{u}\times(\mathbf{v}\times\mathbf{w}))$. Confirm the result by finding $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ and differentiating the result with respect to t.

21. Find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, and κ as functions of t for the helix $\mathbf{r}(t)=a\cos\omega t\mathbf{i}+a\sin\omega t\mathbf{j}+bt\mathbf{k}$.

22. By differentiating $\mathbf{B}=\mathbf{T}\times\mathbf{N}$ with respect to s, show that

$$\frac{d\mathbf{B}}{ds}=\mathbf{T}\times\frac{d\mathbf{N}}{ds},$$

and then by forming the product $\mathbf{N}\times d\mathbf{B}/ds$, show that

$$\mathbf{N}\times\frac{d\mathbf{B}}{ds}=\mathbf{0}.$$

Introduce a constant of proportionality called the **torsion** of the curve Γ at P, which by convention is denoted by $-\tau$, and deduce from this last result that

$$\frac{d\mathbf{B}}{ds}=-\tau\mathbf{N}.$$

Finally, by differentiating $\mathbf{N}=\mathbf{B}\times\mathbf{T}$ with respect to s show that

$$\frac{d\mathbf{N}}{ds}=\tau\mathbf{B}-\kappa\mathbf{T}.$$

The three equations relating the derivatives of $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ with respect to s to $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, κ, and τ found earlier, namely,

$$\frac{d\mathbf{T}}{ds}=\kappa\mathbf{N},\quad\frac{d\mathbf{N}}{ds}=\tau\mathbf{B}-\kappa\mathbf{T},\quad\text{and}\quad\frac{d\mathbf{B}}{ds}=-\tau\mathbf{N},$$

are called the **Frenet–Serret equations**, and they are fundamental to the study of the differential geometry of space curves.

11.2 Integration of Scalar and Vector Functions of a Single Real Variable

As with real functions of a single real variable, a differentiable vector function of a single real variable $\mathbf{F}(t)$ will be called an **antiderivative** of the vector function $\mathbf{f}(t)$ on some interval $a<t<b$ if at each point of the interval $d\mathbf{F}(t)/dt=\mathbf{f}(t)$. Because differentiation of a vector constant yields the null vector $\mathbf{0}$, an antiderivative of $\mathbf{f}$ is only determined up to an arbitrary additive vector constant $\mathbf{C}$. An **indefinite integral** of $\mathbf{f}$ is any antiderivative of $\mathbf{f}$ to which has been added an arbitrary vector constant.

indefinite and definite integrals of vector functions of a single real variable

Indefinite and definite integrals of a vector function of a single real variable

If $\mathbf{F}(t)$ is any antiderivative of $\mathbf{f}(t)$, then an **indefinite integral** of the function $\mathbf{f}$ with respect to t, written $\int\mathbf{f}(t)dt$, is

$$\int\mathbf{f}(t)dt=\mathbf{F}(t)+\mathbf{C},$$

where $\mathbf{C}$ is an arbitrary vector constant.

If $\mathbf{f}(t)=f_1(t)\mathbf{i}+f_2(t)\mathbf{j}+f_3(t)\mathbf{k}$, the indefinite integral of $\mathbf{f}(t)$ is determined by integrating each component of $\mathbf{f}(t)$ with respect to t and combining

the results to give

$$\int f_1(t)dt\mathbf{i} + \int f_2(t)dt\mathbf{j} + \int f_3(t)dt\mathbf{k} = \mathbf{F}(t) + \mathbf{C}.$$

The **definite integral** of $\mathbf{f}(t)$ over the interval $a \le t \le b$ is defined as

$$\int_a^b \mathbf{f}(t)dt = \int_a^b f_1(t)dt\mathbf{i} + \int_a^b f_2(t)dt\mathbf{j} + \int_a^b f_3(t)dt\mathbf{k}.$$

EXAMPLE 11.8

Given that $\mathbf{f}(t) = \sin t\mathbf{i} + (1 - t^2)\mathbf{j} + e^{-t}\mathbf{k}$, find

$$\text{(a)} \int \mathbf{f}(t)dt \quad \text{and} \quad \text{(b)} \int_0^2 \mathbf{f}(t)dt.$$

Solution

(a)

$$\begin{aligned}\int \mathbf{f}(t)dt &= \int \sin t\,dt\mathbf{i} + \int (1-t^2)dt\mathbf{j} + \int e^{-t}\,dt\mathbf{k}\\ &= -\cos t\mathbf{i} + \left(t - \frac{1}{3}t^3\right)\mathbf{j} - e^{-t}\mathbf{k} + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k},\end{aligned}$$

where c_1, c_2, and c_3 are arbitrary real constants, so

$$\int \mathbf{f}(t)dt = -\cos t\mathbf{i} + \left(t - \frac{1}{3}t^3\right)\mathbf{j} - e^{-t}\mathbf{k} + \mathbf{C},$$

where $\mathbf{C}$ is an arbitrary vector constant.

(b)

$$\begin{aligned}\int_0^2 \mathbf{f}(t)dt &= \int_0^2 \sin t\,dt\mathbf{i} + \int_0^2 (1-t^2)dt\mathbf{j} + \int_0^2 e^{-t}dt\mathbf{k}\\ &= (1 - \cos 2)\mathbf{i} - \frac{2}{3}\mathbf{j} + (1 - e^{-2})\mathbf{k}.\end{aligned}$$

It is sometimes necessary to find the length of arc between two points on a curve defined by a vector function of a single real variable. This can be accomplished by making use of result (6), which showed that the rate of change of distance s with respect to t along the curve Γ defined by

$$\mathbf{r}(t) = x_1(t)\mathbf{i} + x_2(t)\mathbf{j} + x_3(t)\mathbf{k}$$

is given by

$$\frac{ds}{dt} = \left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2\right]^{1/2}.$$

arc length along a space curve

Consequently, if the length of arc $s = s(t_2) - s(t_1)$ between the points corresponding to $t = t_1$ and $t = t_2$ is required, where $t_2 > t_1$, integration of this result gives

$$\int_{t_1}^{t_2} \frac{ds}{dt}dt = \int_{t_1}^{t_2} \left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2\right]^{1/2} dt,$$

so the required arc length is given by the definite integral

$$s = s(t_2) - s(t_1) = \int_{t_1}^{t_2} \left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2 \right]^{1/2} dt. \tag{15}$$

EXAMPLE 11.9 Find the length of arc along the helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \alpha t\mathbf{k}$ between the points corresponding to $t = 0$ and $t = 2\pi$, where α is a scalar constant.

Solution Making the identifications $x_1(t) = \cos t$, $x_2(t) = \sin t$, $x_3(t) = \alpha t$, $t_1 = 0$, and $t_2 = 2\pi$, and substituting into (15) gives

$$s = \int_0^{2\pi} [(-\sin t)^2 + (\cos t)^2 + \alpha^2]^{1/2}\, dt$$

$$= \sqrt{1+\alpha^2} \int_0^{2\pi} dt = 2\pi\sqrt{1+\alpha^2}.$$

When $\alpha = 0$ the helix reduces to a circle of unit radius, and as expected s then becomes the circumference 2π of a unit circle. ■

Let the vector $\mathbf{F}(x, y, z)$ be defined along a piecewise smooth space curve Γ along which the arc length is s, and let Γ extend from the point $\mathbf{r}_1$ at which $s = s_1$ to the point $\mathbf{r}_2$ at which $s = s_2$. Then, if $\mathbf{T}(s)$ is the unit tangent vector to Γ at arc length s, an expression of the form

$$I = \int_{s_1}^{s_2} \mathbf{F} \cdot \mathbf{T}\, ds$$

scalar line integrals

is called a **line integral** of $\mathbf{F}$, or more precisely, the **scalar line integral** of $\mathbf{F}$ along the space curve Γ. It follows from (8) that $\mathbf{T}ds = d\mathbf{r}$, so the line integral of $\mathbf{F}$ along Γ can be written in the simpler form

$$I = \int_{s_1}^{s_2} \mathbf{F} \cdot d\mathbf{r}. \tag{16}$$

Integrals of this type have many applications, two of the most important of which are described in what follows. The first application is to mechanics, where when a constant force $\mathbf{F}$ moves its point of application a distance d along a straight line L, the **work** that is done by the force is $W = f_L d$, where f_L is the component of $\mathbf{F}$ along the line L. To find the work done by a variable force $\mathbf{F}(t)$ as it moves its point of application along a parametrically defined curve Γ, it is necessary to generalize this simple result by appealing to the notion of a line integral along the space curve Γ.

If the vector differential along Γ is denoted by $d\mathbf{r}$, its length $|d\mathbf{r}| = dr$, so the unit vector $\mathbf{T}$ in the direction $d\mathbf{r}$ will be $\mathbf{T} = d\mathbf{r}/dr$. Consequently, the component of force $\mathbf{F}$ in the direction of $d\mathbf{r}$ is given by $\mathbf{F} \cdot \mathbf{T} = (\mathbf{F} \cdot d\mathbf{r})/dr$, so the element of

work dW performed by the force in moving its point of application along $d\mathbf{r}$ will be

$$dW = \mathbf{F} \cdot \left(\frac{d\mathbf{r}}{dr}\right) dr = \mathbf{F} \cdot d\mathbf{r}.$$

Integration of this result shows the work performed by the force in moving its point of application along Γ from $\mathbf{r} = \mathbf{r}_1$ to $\mathbf{r} = \mathbf{r}_2$, corresponding to $s = s_1$ and $s = s_2$, respectively, is given by the line integral

$$W = \int_{s_1}^{s_2} \mathbf{F} \cdot d\mathbf{r}. \tag{17}$$

When $\mathbf{r} = \mathbf{r}(t)$ is known as a function of t, but t is not the arc length s along Γ, and integration is between $\mathbf{r} = \mathbf{r}(t_1)$ and $\mathbf{r} = \mathbf{r}(t_2)$, $d\mathbf{r} = (d\mathbf{r}/dt)dt$ and (17) becomes

$$W = \int_{t(s_1)}^{t(s_2)} \mathbf{F}(\mathbf{r}(t)) \cdot (d\mathbf{r}/dt)dt. \tag{18}$$

Integrals of this type arise when particles move in a gravitational field or a charged particle moves in an electric field. The sign of W depends on the direction of integration, so reversing its direction changes the sign of W. Work is done by the vector field $\mathbf{F}$ when W is positive, and work is recovered from the field when W is negative.

For the second example we consider the case of fluid mechanics and identify $\mathbf{F}$ with the fluid velocity vector $\mathbf{q}$. In this case a line integral of the form (16) is called the **flow** of the fluid along Γ, because $d\mathbf{r} = (d\mathbf{r}/ds)ds = \mathbf{T}ds$, where $\mathbf{T}$ is the unit tangent along Γ, so that $\mathbf{q} \cdot \mathbf{T}$ is the component of the flow along Γ. The **circulation** k of fluid is defined as the flow around a *closed* curve Γ, so it is given by

circulation and irrotational flow

$$k = \oint_\Gamma \mathbf{q} \cdot d\mathbf{r} = \oint_\Gamma \mathbf{q} \cdot \mathbf{T}\, ds, \tag{19}$$

where the symbol $\oint_\Gamma$ is used to indicate that the line integral of $\mathbf{q} \cdot d\mathbf{r}$ is taken *once* around the closed curve Γ.

In fluid mechanics the circulation k describes an important characteristic of the fluid motion, and it can be seen from (19) that reversing the direction of integration around Γ reverses the sign of $\mathbf{T}$, and so leads to a reversal of the sign of the circulation. The fundamental class of fluid flow in which there is zero circulation around every simple closed curve Γ, so that $k \equiv 0$, is called **irrotational** flow.

In general, the line integral (16) depends not only on $\mathbf{F}$ and the end points of integration, but also on the path Γ along which the integral is evaluated. The method of evaluating line integrals, and the fact that they usually depend on the path, is illustrated in the next example.

EXAMPLE 11.10

Find the line integral of $\mathbf{F} = -yz^2\mathbf{i} + xz^2\mathbf{j} + yz\mathbf{k}$ (a) along the helix Γ given by $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$ from $t = 0$ to $t = 2\pi$, and (b) along the straight line path γ joining the points $\mathbf{r}(0)$ to $\mathbf{r}(2\pi)$.

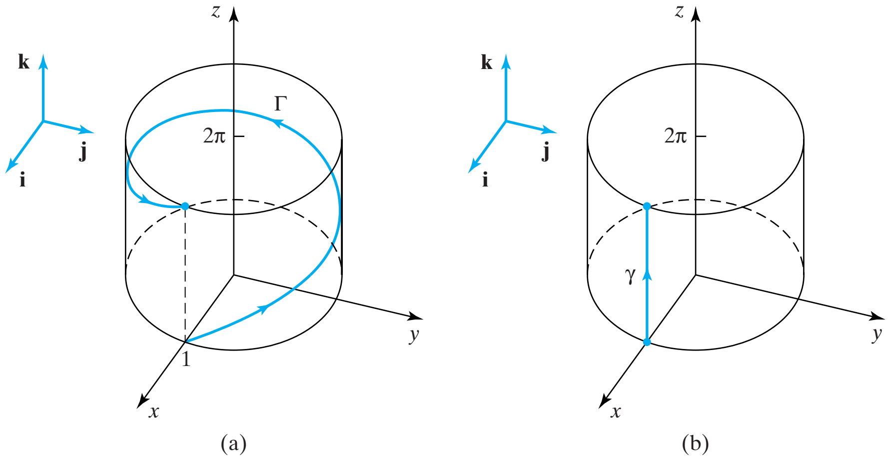


FIGURE 11.5 (a) The helix Γ. (b) The straight line path γ.

Solution

(a) The helix Γ is shown in Fig. 11.5(a).
Differentiation of $\mathbf{r}(t)$ gives

$$\frac{d\mathbf{r}}{dt} = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k},$$

but on the helix $x = \cos t$, $y = \sin t$, and $z = t$, so in the line integral along Γ the general vector-valued function $\mathbf{F}$ becomes the vector function of the single real variable t given by $\mathbf{F}(t) = -t^2 \sin t\mathbf{i} + t^2 \cos t\mathbf{j} + t \sin t\mathbf{k}$. As a result,

$$\begin{aligned}\mathbf{F} \cdot d\mathbf{r} &= (-t^2 \sin t\mathbf{i} + t^2 \cos t\mathbf{j} + t \sin t\mathbf{k}) \cdot (-\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k})dt \\ &= (t^2 + t \sin t)dt,\end{aligned}$$

and so the required line integral is

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (t^2 + t \sin t)dt = \frac{8}{3}\pi^3 - 2\pi.$$

(b) The straight line path γ shown in Fig. 11.5(b) joins the points $\mathbf{r}(0) = \mathbf{i}$ and $\mathbf{r}(2\pi) = \mathbf{i} + 2\pi\mathbf{k}$, so in terms of the parameter t its vector equation can be written $\mathbf{r}(t) = \mathbf{i} + t\mathbf{k}$ with $0 \le t \le 2\pi$. This shows that on the path γ we have $x = 1$, $y = 0$, and $z = t$, and $d\mathbf{r} = dt\mathbf{k}$.

Consequently, on γ the vector-valued function $\mathbf{F}$ becomes $\mathbf{F} = t^2\mathbf{j}$, and so

$$\mathbf{F} \cdot d\mathbf{r} = t^2\mathbf{j} \cdot (dt\mathbf{k}) = 0,$$

showing that

$$\int_\gamma \mathbf{F} \cdot d\mathbf{r} = 0.$$

In the next section, after the introduction of the *gradient* of a function, we will find a condition to be satisfied by $\mathbf{F}$ in order that the line integral in (16) is independent of the path Γ, and so depends only on $\mathbf{F}$ and the end points of the integration.

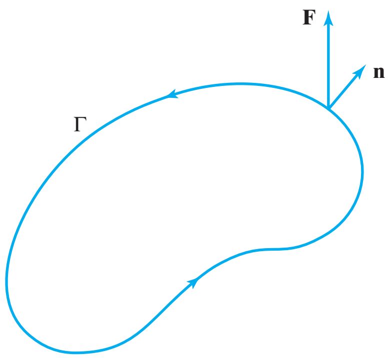


FIGURE 11.6 $\Phi_\Gamma = \oint_\Gamma \mathbf{F}\cdot\mathbf{n}\,ds$ is the flux of $\mathbf{F}$ across Γ.

As a final example of an application of line integrals we determine the *flux* of a vector $\mathbf{F}(x, y)$ across a closed two-dimensional smooth curve Γ in the (x, y)-plane. If $\mathbf{n}$ is a unit vector normal to Γ that is directed *outward* from Γ, as shown in Fig. 11.6, the **flux** Φ_Γ across the curve Γ is defined as the line integral

$$\Phi_\Gamma = \int_\Gamma \mathbf{F}\cdot\mathbf{n}\,ds,$$

the flux of a vector across a plane curve

where s is the arc length around Γ and integration is in the *counterclockwise* sense around Γ. As $\mathbf{F}\cdot\mathbf{n}$ is the component of $\mathbf{F}$ in the direction of the outward drawn normal to Γ, the flux Φ_Γ is seen to measure the total amount of the normal component of $\mathbf{F}$ that crosses the curve Γ.

For a physical illustration of the meaning of flux, let us consider a long block of metal with its axis in the z-direction in which there is a steady-state temperature distribution that is only a function of x and y. This means that the temperature distribution is the same in every plane $z =$ constant. Let us now consider a cylindrical region in the block of unit height and cross-section Γ with its axis in the z-direction. Then if $\mathbf{F}$ is identified with a heat flow vector $\mathbf{h}(x, y)$, the flux Φ_Γ is the amount of the heat that crosses the curved walls of this cylinder in Fig. 11.7 in a unit time. If $\Phi_\Gamma > 0$ there is a net *outflow* of heat from the region bounded by Γ, and if $\Phi_\Gamma < 0$ there is a net *inflow* of heat into the region. When $\Phi_\Gamma = 0$ the amount of heat in the region remains constant.

In two space dimensions it is important to recognize the difference between the circulation and flux of $\mathbf{F}$ in relation to the curve Γ. Whereas the determination of the *circulation* of $\mathbf{F}$ involves the line integral of the component of $\mathbf{F}$ *along* the tangent to curve Γ with respect to the arc length s, the *flux* of $\mathbf{F}$ involves the line integral of the component of $\mathbf{F}$ normal to (*across*) the curve Γ with respect to the arc length.

To determine the flux we proceed as follows. Let $\mathbf{F}(x, y) = f_1(x, y)\mathbf{i} + f_2(x, y)\mathbf{j}$ and Γ have the equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. Then, as integration around Γ is in the counterclockwise sense, we see from Fig. 11.6 that if $\mathbf{T}$ is the unit tangent to Γ, then $\mathbf{n} = \mathbf{T}\times\mathbf{k}$. As $\mathbf{T} = (dx/ds)\mathbf{i} + (dy/ds)\mathbf{j}$, it follows that

$$\begin{aligned}\mathbf{n} = \mathbf{T}\times\mathbf{k} &= [(dx/ds)\mathbf{i} + (dy/ds)\mathbf{j}]\times\mathbf{k}\\ &= (dy/ds)\mathbf{i} - (dx/ds)\mathbf{j},\end{aligned}$$

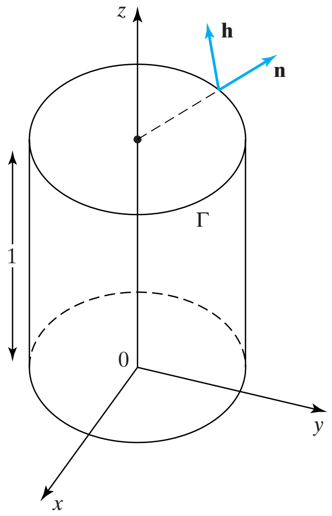


FIGURE 11.7 A cylinder of unit height and cross-section Γ with its axis in the z-direction.

and so

$$\Phi_\Gamma = \int_\Gamma \mathbf{F} \cdot \mathbf{n}\, ds = \int_\Gamma (f_1(x, y)\mathbf{i} + f_2(x, y)\mathbf{j}) \cdot ((dy/ds)\mathbf{i} - (dx/ds)\mathbf{j})ds$$

$$= \int_\Gamma f_1(x, y)dy - f_2(x, y)dx.$$

EXAMPLE 11.11 Find the flux of $\mathbf{F} = (2x + y)\mathbf{i} + (y - x)\mathbf{j}$ across the ellipse with the equation $x^2/a^2 + y^2/b^2 = 1$.

Solution By setting $x = a\cos t$ and $y = b\sin t$ and restricting t to the interval $0 \le t \le 2\pi$, the ellipse is traversed once in the counterclockwise sense as required. As $dx = -a\sin t\, dt$ and $dy = b\cos t\, dt$, substitution into the expression for Φ_Γ gives

$$\Phi_\Gamma = \int_0^{2\pi} [(2a\cos t + b\sin t)b\cos t - (b\sin t - a\cos t)(-a\sin t)]\, dt = 3ab\pi. \quad \blacksquare$$

Finally we define a different integral called a *vector line integral* of **F**. To do this we let a curve Γ have the vector equation

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad \text{for } a \le t \le b$$

and introduce a general vector function $\mathbf{F} = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$ defined along the curve Γ. Then the **vector line integral** of **F** along Γ from $t = a$ to $t = b$ is defined as

$$\int_a^b \mathbf{F}\, dt = \mathbf{i}\int_a^b F_1(t)dt + \mathbf{j}\int_a^b F_2(t)dt + \mathbf{k}\int_a^b F_3(t)dt, \tag{20}$$

where $F_i(t) = F_i(x(t), y(t), z(t))$, for $i = 1, 2, 3$.

EXAMPLE 11.12 Find the vector line integral of the vector function $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + z\mathbf{k}$ along the curve $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + t\mathbf{k}$ over the interval $0 \le t \le \pi$.

Solution

$$\int_0^{\pi} \mathbf{F}\,dt = \mathbf{i}\int_0^{\pi} at\cos t\,dt + \mathbf{j}\int_0^{\pi} at\sin t\,dt + \mathbf{k}\int_0^{\pi} t\,dt = -2a\mathbf{i} + \pi a\mathbf{j} + \frac{1}{2}\pi^2\mathbf{k}.$$

Summary

Indefinite and definite integrals of vector functions of a single real variable have been defined and illustrated by example. The scalar line integral of a vector $\mathbf{F}(x, y, z)$ has been defined and its application illustrated by considering the work done by a force as it moves along a space curve between two fixed points. The line integral has also been applied to fluid flow and used to define the circulation of the fluid, and the related concept of an irrotational flow for which the circulation around any closed curve in the fluid is zero. Finally, the flux of a vector across a plane curve has been defined.

EXERCISES 11.2

In Exercises 1 through 4 find the required indefinite and definite integrals.

1. (a) $\int (t\sin t\mathbf{i} + 3t^2\mathbf{j} - 3t\mathbf{k})dt.$

(b) $\int_0^2 (\ln(1+3t)\mathbf{i} + (t^3 - 2t)\mathbf{j} + te^t\mathbf{k})dt.$

2. (a) $\int (\cosh^2 t\mathbf{i} + 2\sin^2 2t\mathbf{j} + \mathbf{k})dt.$

(b) $\int_0^2 ((1+t^2)^{-1}\mathbf{i} - t\sin t\mathbf{j} - (1-3t^2)\mathbf{k})dt.$

3. (a) $\int (\cos^2 3t\mathbf{i} + \sin^2 t\mathbf{j} + t\mathbf{k})dt.$

(b) $\int_0^{\pi} ((1+3t^2)\mathbf{i} + \cos 4t\mathbf{j} + \sin 3t\mathbf{k})dt.$

4. (a) $\int (t(1+t)^{-1}\mathbf{i} + \sec^2 3t\mathbf{j} + (t^2 - 4)\mathbf{k})dt.$

(b) $\int_0^4 (t(1+3t^2)^{-1}\mathbf{i} + (1+t^2)^{1/2}\mathbf{j} + t^2e^{-t}\mathbf{k})dt.$

5. Find the arc length along the circular helix $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + \alpha t\mathbf{k}$ between the points corresponding to $t = \pi$ and $t = 3\pi/2$.

6. Find the arc length along the curve $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$ between the points corresponding to $t = 0$ and $t = 2\pi$.

7. Given the vector valued function $\mathbf{F} = -z\mathbf{i} + x\mathbf{j} - y\mathbf{k}$, find the scalar line integral of $\mathbf{F}$ along the space curve $\mathbf{r}(t) = \sin t\mathbf{i} - \cos t\mathbf{j} + e^t\mathbf{k}$ between the points on the curve corresponding to $t = 0$ and $t = \pi/2$.

8. Given the vector valued function $\mathbf{F} = 2y\mathbf{i} + x^2\mathbf{j} - 3z\mathbf{k}$, find the line integral of $\mathbf{F}$ along the space curve $\mathbf{r}(t) = t\mathbf{i} + (1+2t^3)\mathbf{j} + t^2\mathbf{k}$ between the points on the curve corresponding to $t = 1$ and $t = 3$.

9. Let $\mathbf{F}$ be the vector-valued function $\mathbf{F} = -x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that the line integrals of $\mathbf{F}$ along the helix $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + t\mathbf{k}$ between the points on the helix corresponding to $t = 0$ and $t = 2\pi$ and along the straight line path joining the points $\mathbf{r}(0)$ to $\mathbf{r}(2\pi)$ are the same.

10. Let $\mathbf{F}$ be the vector-valued function $\mathbf{F} = 2xy^2z\mathbf{i} + 2x^2yz\mathbf{j} + x^2y^2\mathbf{k}$. Find the line integral of $\mathbf{F}$ along the straight line Γ with the equation $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$ between the points corresponding to $t = 0$ and $t = 1$. Let γ be the path formed by the straight line segments joining the points $PQRS$, in this order, where P is the point $\mathbf{r} = \mathbf{0}$, Q is the point $\mathbf{r} = \mathbf{i}$, R is the point $\mathbf{r} = \mathbf{i} + 2\mathbf{j}$, and S is the point $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Find the line integral of $\mathbf{F}$ along γ from P to S, and hence show that it has the same value as the integral along Γ.

11. The velocity vector in a two-dimensional fluid flow is $\mathbf{v} = y\mathbf{i} + x^2y\mathbf{j}$. Find the circulation (a) around the ellipse $x^2 + \frac{1}{4}y^2 = 1$ and (b) around the unit circle $x^2 + y^2 = 1$, and hence show the flow is *not* irrotational.

12. The velocity vector in a two-dimensional fluid flow is $\mathbf{v} = (2x + 3y^2)\mathbf{i} + 6x\mathbf{j}$. Show that there is zero circulation around all the circles $(x-a)^2 + (y-b)^2 = c^2$, where a, b, and $c > 0$ are arbitrary real numbers. Is it correct to say this proves that the flow is irrotational? Give reasons justifying your answer.

13. Find the flux of $\mathbf{F} = (3x + 2y)\mathbf{i} + (2x - y)\mathbf{j}$ across the circle $x^2 + y^2 = 4$.

11.3 Directional Derivatives and the Gradient Operator

Consider a scalar function $w = f(x, y, z)$ with continuous first order partial derivatives with respect to x, y, and z that is defined in some region D of space, and let a space curve Γ in D have the parametric equations $x = x(t)$, $y = y(t)$, and $z = z(t)$. Then from the chain rule

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}, \tag{21}$$

and it is seen from this that dw/dt can be interpreted as the scalar product of the two vectors

$$\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \quad \text{and} \quad \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}.$$

The first vector, denoted by

$$\text{grad } f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}, \tag{22}$$

the gradient of a scalar function of position

is called the **gradient** of the scalar function f expressed in terms of cartesian coordinates, whereas from Section 11.1 the second vector

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \tag{23}$$

is seen to be a vector that is tangent to the space curve Γ. Consequently, dw/dt is the scalar product of grad f and $d\mathbf{r}/dt$ at the point $x = x(t)$, $y = y(t)$, and $z = z(t)$ for any given value of t.

Another notation for grad f that is also used is

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}, \tag{24}$$

where the symbol ∇f is either read “del f” or “grad f.” In this notation, the **vector operator**

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{25}$$

is the **gradient operator** expressed in terms of cartesian coordinates, and if ϕ is a suitably differentiable scalar function of x, y, and z, it is to be understood that

$$\nabla\phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}. \tag{26}$$

Let us now introduce the unit vector $\mathbf{v}$ defined as

$$\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}, \tag{27}$$

where l, m, and n are the direction cosines of the tangent to the space curve Γ in (23), so that

$$l = \frac{dx}{dt}\Big/\left|\frac{d\mathbf{r}}{dt}\right|, \quad m = \frac{dy}{dt}\Big/\left|\frac{d\mathbf{r}}{dt}\right|, \quad n = \frac{dz}{dt}\Big/\left|\frac{d\mathbf{r}}{dt}\right|, \tag{28}$$

with

$$\left|\frac{d\mathbf{r}}{dt}\right| = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]^{1/2}. \tag{29}$$

Then as the scalar product of a vector $\mathbf{F}$ and the unit vector $\mathbf{v}$ is the *projection* of $\mathbf{F}$ in the direction $\mathbf{v}$, it follows at once that

$$D_v f = \mathbf{v} \cdot \operatorname{grad} f = l\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y} + n\frac{\partial f}{\partial z} \tag{30}$$

the directional derivative and its properties

is the **directional derivative** of f in the direction $\mathbf{v}$. This last result has meaning irrespective of whether $\mathbf{v}$ is tangent to a space curve, so from now on $\mathbf{v}$ can be taken to be an arbitrary unit vector in space.

The directional derivative $D_v f$ can be interpreted in terms of the ordinary operation of differentiation by considering Fig. 11.8. In the diagram, a straight line T in space in the direction of a given vector $\mathbf{v}$ passes through a fixed point P, and Q is a general point on line T at a distance s from P. The directional derivative $D_v f$ is then given by

$$D_v f = \frac{df}{dv} = \lim_{s\to 0} \frac{f(Q) - f(P)}{s}. \tag{31}$$

In the two-dimensional case in the (x, y)-plane, the directional derivative defined in (30) simplifies to

$$D_v f = \mathbf{v} \cdot \operatorname{grad} f = l\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y}, \tag{32}$$

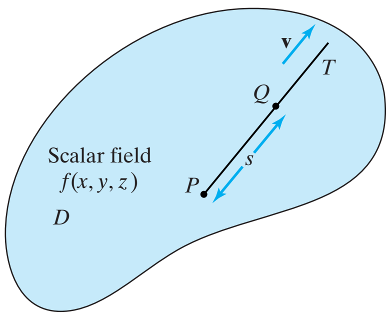


FIGURE 11.8 The directional derivative $D_v f$.

where now the unit vector $\mathbf{v} = l\mathbf{i} + m\mathbf{j}$, with $l^2 + m^2 = 1$, and the grad f in (22) simplifies to

$$\text{grad } f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}, \tag{33}$$

where again the unit vector $\mathbf{v} = l\mathbf{i} + m\mathbf{j}$, with $l^2 + m^2 = 1$.

EXAMPLE 11.13

Find the directional derivative of $f = x^2 + 3y^2 + 2z^2$ in the direction of the vector $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, and determine its value at the point $(1, -3, 2)$.

Solution grad $f = 2x\mathbf{i} + 6y\mathbf{j} + 4z\mathbf{k}$ and the unit vector in the required direction is $v = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$, and so the required directional derivative is

$$D_v f = \left(\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) \cdot (2x\mathbf{i} + 6y\mathbf{j} + 4z\mathbf{k}),$$

and so

$$D_v f = \frac{4}{3}x - 2y - \frac{8}{3}z.$$

This shows that the directional derivative $D_v f$ at the point $(1, -3, 2)$ is

$$D_v f(1, -3, 2) = \frac{4}{3} + 6 - \frac{16}{3} = 2.$$

■

Inspection of definition (30) shows immediately that $D_v f$, which is the rate of change of f in the direction $\mathbf{v}$, must take its greatest value when $\mathbf{v}$ is in the direction of grad f, its smallest value when $\mathbf{v}$ and grad f are oppositely directed, and the value zero when $\mathbf{v}$ and grad f are orthogonal. These simple properties of a directional derivative are sufficiently important for them to be recorded separately in the following form.

Properties of directional derivatives

1. The most rapid increase of a differentiable function $f(x, y, z)$ at a point P in space occurs in the direction of the vector $\mathbf{v}_P = \text{grad } f(P)$. The directional derivative at P is then given by

$$D_v f(P) = |\text{grad } f(P)| = \left((\partial f/\partial x)_P^2 + (\partial f/\partial y)_P^2 + (\partial f/\partial z)_P^2\right)^{1/2}.$$

2. The most rapid decrease of a differentiable function $f(x, y, z)$ at a point P in space occurs when the vector $\mathbf{v}_P$ just defined in 1 and grad f are oppositely directed, so that $\mathbf{v}_P = -\text{grad } f(P)$. The directional derivative at P is then the *negative* of the result in 1 and so is given by

$$\begin{aligned} D_v f(P) &= -|\text{grad } f(P)| \\ &= -\left((\partial f/\partial x)_P^2 + (\partial f/\partial y)_P^2 + (\partial f/\partial z)_P^2\right)^{1/2}. \end{aligned}$$

3. There is a zero local rate of change of a differentiable function $f(x, y, z)$ at a point P in space in the direction of any vector $\mathbf{v}_P$ that is orthogonal to grad f at P, so that $\mathbf{v}_P \cdot \text{grad } f(P) = 0$.

When a scalar function f defined over a region D of space is suitably differentiable, the vector-valued function grad f defines a *vector field* over D in terms of the *scalar field* defined by f. The next theorem establishes the result of performing the gradient operation on combinations of scalar functions.

THEOREM 11.2

properties of the gradient operator

Rules for the gradient operator Let the gradients of f and g be defined over a region D. Then the gradient operator has the following properties.

(i) Gradient of a constant multiple of f:

$$\text{grad}\,(cf) = c\,\text{grad}\,f; \qquad (c \text{ a scalar constant})$$

(ii) Gradient of a sum or difference of functions:

$$\text{grad}\,(f \pm g) = \text{grad}\,f \pm \text{grad}\,g;$$

(iii) Gradient of a product of functions:

$$\text{grad}\,(fg) = f\,\text{grad}\,g + g\,\text{grad}\,f;$$

(iv) Gradient of a quotient of functions:

$$\text{grad}\left(\frac{f}{g}\right) = (g\,\text{grad}\,f - f\,\text{grad}\,g)/g^2 \qquad (g \neq 0).$$

Proof These results all follow by applying the usual rules for partial differentiation to each component of the gradient function on the left, and then recombining the results to obtain the expression on the right. To illustrate the form of argument involved, we prove result (iii) concerning the gradient of a product of functions. By definition,

$$\begin{aligned}
\text{grad}(fg) &= \frac{\partial(fg)}{\partial x}\mathbf{i} + \frac{\partial(fg)}{\partial y}\mathbf{j} + \frac{\partial(fg)}{\partial z}\mathbf{k} \\
&= \left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right)\mathbf{i} + \left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right)\mathbf{j} + \left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right)\mathbf{k} \\
&= f\,\text{grad}\,g + g\,\text{grad}\,f.
\end{aligned}$$

■

A simple application of the gradient of a function involves the determination of the tangent plane to the surface S defined by the function $f(x, y, z) = \text{constant}$ at a point $P_0(x_0, y_0, z_0)$ on the surface S.

Define the function $w = f(x, y, z) - c$, where $c = \text{constant}$, so that the surface S then has the equation $w = 0$. Let any space curve Γ in the surface S have the parametric equations

$$x = x(t), \quad y = y(t), \quad \text{and} \quad z = z(t).$$

Then differentiation of $w = f(x, y, z) - c$ with respect to t gives

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt},$$

but on S the function $w \equiv 0$, so this reduces to

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0.$$

This result shows that any curve Γ in S must be orthogonal to grad f, and so at every point P of the surface S the vector grad f is normal to the surface. The vector equation of a plane with normal $\mathbf{n}$ containing the point P_0 with position vector $\mathbf{r}_0$ is

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0,$$

where $\mathbf{r}$ is the position vector of an arbitrary point on the plane. If we set $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$, and identify $\mathbf{n}$ with grad f at P_0, where

$$\text{grad } f(P_0) = \left(\frac{\partial f}{\partial x}\right)_{P_0} \mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \mathbf{j} + \left(\frac{\partial f}{\partial z}\right)_{P_0} \mathbf{k},$$

the required tangent plane to the surface at $P_0(x_0, y_0, z_0)$ is seen to be given by

$$(x - x_0)\left(\frac{\partial f}{\partial x}\right)_{P_0} + (y - y_0)\left(\frac{\partial f}{\partial y}\right)_{P_0} + (z - z_0)\left(\frac{\partial f}{\partial z}\right)_{P_0} = 0 \tag{34}$$

EXAMPLE 11.14 Find the tangent plane at the point $(2, -1, 3)$ on the sphere

$$(x - 1)^2 + (y + 2)^2 + (z - 4)^2 = 3.$$

Solution It is first necessary to check that the point $(2, -1, 3)$ does actually lie on the sphere, and this is confirmed by showing that $x = 2$, $y = -1$, and $z = 3$ satisfies the equation of the sphere. Writing $f = (x - 1)^2 + (y + 2)^2 + (z - 4)^2$, we find that $\partial f/\partial x = 2x$, $\partial f/\partial y = 2y$, and $\partial f/\partial z = 2z$, so that $(\partial f/\partial x)_{(2,-1,3)} = 4$, $(\partial f/\partial y)_{(2,-1,3)} = -2$, and $(\partial f/\partial z)_{(2,-1,3)} = 6$. Substitution into (34) shows that the equation of the tangent plane to the sphere at the point $(2, -1, 3)$ is

$$4(x - 2) - 2(y + 1) + 6(z - 3) = 0,$$

and after simplification this reduces to

$$4x - 2y + 6z = 28.$$

In applications, the geometry of a problem often makes it necessary to express the gradient operator in terms of different coordinate systems. The coordinate systems that occur most frequently as a result of formulating problems involving either a cylindrical or a spherical geometry are the cylindrical polar coordinate system (r, θ, z) illustrated in Fig. 11.9a and the spherical polar coordinate system (r, θ, ϕ) illustrated in Fig. 11.9b, and shown in a different form in Fig. 1.15.

Consideration of the geometry of Figs. 11.9a,b establishes that the connection between these coordinate systems and the cartesian coordinates (x, y, z) is given by:

Cylindrical polar coordinates (r, θ, z)

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \tag{35}$$

Spherical polar coordinates (r, θ, ϕ)

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta. \tag{36}$$

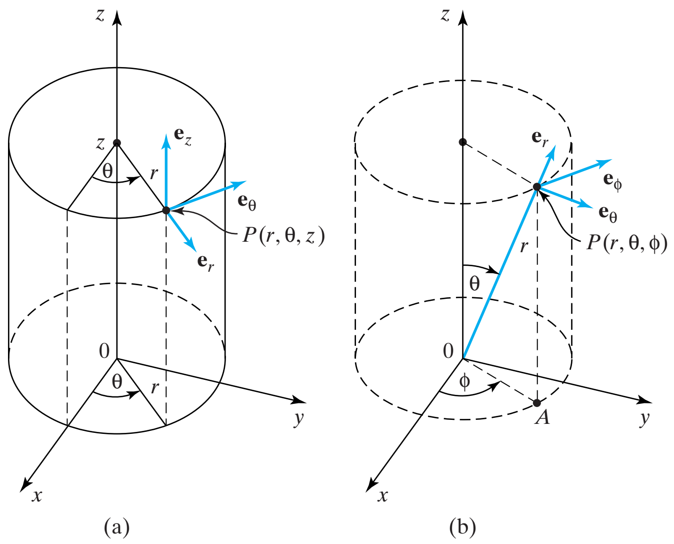


FIGURE 11.9 (a) Cylindrical polar coordinates. (b) Spherical polar coordinates.

The forms taken by grad f in cylindrical and spherical polar coordinates are given next for reference, though the derivation of these results together with related results in terms of general orthogonal curvilinear coordinates will be postponed until Section 11.6.

gradient operator in cylindrical polar coordinates

grad f in cylindrical polar coordinates (r, θ, z)

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z, \tag{37}$$

where $\mathbf{e}_r$ is a unit vector parellel to the (x, y)-plane along the radial line r, $\mathbf{e}_\theta$ is a unit vector in the (x, y)-plane normal to $\mathbf{e}_r$ in the direction of increasing θ, and $\mathbf{e}_z$ is a unit vector in the positive z-direction as shown in Fig. 11.9a, so that $\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_z$.

grad f in spherical polar coordinates (r, θ, ϕ)

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi, \tag{38}$$

where $\mathbf{e}_r$ is a unit vector along the radial line r, $\mathbf{e}_\theta$ is a unit vector in the direction of increasing θ, and $\mathbf{e}_\phi$ is a unit vector in the direction of increasing ϕ that is normal to the plane containing $\mathbf{e}_r$ and $\mathbf{e}_\theta$, as shown in Fig. 11.9b, so that $\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_\phi$.

Notations for cylindrical and spherical polar coordinates are not uniform, so when consulting other works it is advisable to check the notation and conventions that are in use. This is particularly important in the case of spherical polar coordinates, where the r used here is sometimes replaced by ρ, with r then used to denote the distance OA in Fig. 11.9b; in addition, the symbols θ and ϕ are often interchanged.

Summary

The gradient of a scalar function of position is a vector, and it has been defined and used to define the concept of a directional derivative. The properties of directional derivatives have been established and the gradient operator has been used to determine the tangent plane to a sphere at a given point on its surface. For future use, the gradient operator has been expressed in terms of both cylindrical and spherical polar coordinates.

EXERCISES 11.3

In Exercises 1 through 8 find the derivative of the scalar function f in the direction of the vector v and find its value at the point P.

1. $f = x\sin y + y\cos x$, with $v = \mathbf{i} + 2\mathbf{j}$ and P the point $(\pi/4, 0)$.
2. $f = x\sinh(x + 2y)$, with $v = 3\mathbf{i} - \mathbf{j}$ and P the point $(1, -2)$.
3. $f = xe^{xy} + 2x - y$, with $v = \mathbf{i} + 4\mathbf{j}$ and P the point $(-2, 1)$.
4. $f = \ln(x + 2y^2)$, with $v = -\mathbf{i} + 2\mathbf{j}$ and P the point $(1, 3)$.
5. $f = \sin(xy) + e^{3xz}$, with $v = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and P the point $(1, \pi/4, 1)$.
6. $f = (x^2y + z)^{1/2}$, with $v = \mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ and P the point $(2, -3, 1)$.
7. $f = \sinh(xy^2z + 3y)$, with $v = 2\mathbf{i} + \mathbf{k}$ and P the point $(1, -2, 2)$.
8. $f = (xz^2 + 3y)^{-1}$, with $v = -3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and P the point $(1, -1, 1)$.
9. Prove result (iv) in Theorem 11.2.
10. Use result (iv) in Theorem 11.2 to find grad (f/g) given that $f = ye^{xy} + z$ and $g = xyz^2 + 1$, and confirm the result by direct calculation.

In Exercises 11 through 14 find grad f and evaluate it at the point P.

11. $f = x^2 + 3xyz - yz^2$, with P the point $(1, 3, -1)$.
12. $f = (x^2 + 2y^2 + 4z^2)^{-1}$, with P the point $(1, 2, 1)$.
13. $f = \exp(xy + 2yz - 3xz)$, with P the point $(1, 0, 2)$.
14. $f = (x^2 + yz + 3z^2)^{1/2}$, with P the point $(1, -1, 2)$.
15. Derive the cartesian form of the equation of the straight line that is normal to the curve $f(x, y) = \text{constant}$ at a point (x_0, y_0) on the curve.
16. Derive the cartesian form of the equation of the tangent line to the curve $f(x, y) = \text{constant}$ at a point (x_0, y_0) on the curve.
17. Find the equation of the tangent plane to the surface $x^3 + 3xy + z^2 = 11$ at the point on the surface $(1, 2, 2)$.
18. Find the equation of the tangent plane to the surface $\sin(xy) + 2\cos(yz) + 3x = 4$ at the point on the surface $(1, \pi/2, 1)$.
19. Derive the vector equation of the straight line that is normal to the surface $f(x, y, z) = \text{constant}$ at a point with position vector $\mathbf{r}_0$ on the surface.
20. If two surfaces $f(x, y, z) = \text{constant}$ and $g(x, y, z) = \text{Constant}$ intersect at a point with position vector $\mathbf{r}_0$, find a vector that is tangent to their curve of intersection of the two surfaces at $\mathbf{r}_0$.
21. Find grad f, given that $f(r, \theta, z) = r^2\sin\theta + rz^2 + 1$.
22. Find grad f, given that $f(r, \phi, \theta) = r\sin\theta\cos\phi + \sin^2\phi$.
23. If $\mathbf{F} = \text{grad}\, f$, prove that
$$\text{grad}(f^n) = nf^{n-1}\mathbf{F}.$$
Use the result to show that when $f = r$ is the distance of a point $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ from the origin, then
$$\text{grad}\, r = \hat{\mathbf{r}} \quad \text{and} \quad \text{grad}\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3},$$
where $\hat{\mathbf{r}}$ is the unit vector in the direction of $\mathbf{r}$, so $\hat{\mathbf{r}} = \mathbf{r}/r$.

11.4 Conservative Fields and Potential Functions

conservative fields and path invariance

Let us reconsider the line integral $\int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r}$ along a path Γ joining the two points $\mathbf{r}_1$ and $\mathbf{r}_2$ in a region D of space. If the value of this line integral is *independent* of the choice of path Γ in D, the vector field $\mathbf{F}$ is called a **conservative field**. The name *conservative* comes from mechanics, where it refers to the study of dynamics in which dissipative effects such as friction can be ignored, so that the sum of the kinetic and potential energy in a system remains constant (is *conserved*), though conservative fields of different types play key roles throughout physics and engineering.

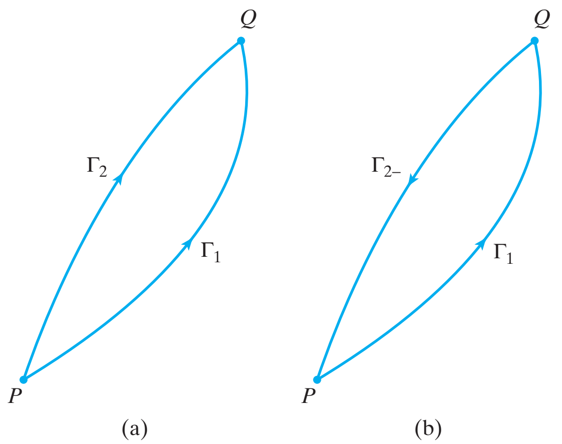


FIGURE 11.10 (a) The two paths Γ_1 and Γ_2. (b) The loop containing P and Q.

The next theorem shows that the definition of a conservative field in terms of the independence of the line integral of the path from $\mathbf{r}_1$ to $\mathbf{r}_2$ is equivalent to the vanishing of the line integral of a conservative field around any closed loop in D.

THEOREM 11.3

Path invariance and integrals around loops If $\mathbf{F}$ is a conservative field in a region D, then $\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = 0$ around every closed loop Γ in D and, conversely, if $\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = 0$ around every closed loop Γ in region D, then $\mathbf{F}$ is a conservative field in D.

Proof The proof of this result is straightforward, and it involves two steps. One is to show that if $\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = 0$ around every closed loop Γ in D, then the field is conservative, and the other involves showing that the converse result is true.

STEP 1 Let the points P and Q shown in Fig. 11.10(a) be any two points in a region D throughout which $\mathbf{F}$ is a conservative field, and let Γ_1 and Γ_2 be any two paths in D connecting P to Q.

As $\mathbf{F}$ is a conservative field, by definition

$$\int_{\Gamma_1} \mathbf{F}\cdot d\mathbf{r} = \int_{\Gamma_2} \mathbf{F}\cdot d\mathbf{r} \quad \text{and so} \quad \int_{\Gamma_1} \mathbf{F}\cdot d\mathbf{r} - \int_{\Gamma_2} \mathbf{F}\cdot d\mathbf{r} = 0.$$

If we reverse the direction of integration in the second integral, thereby changing its sign, and indicate the path from Q to P by Γ_{2-}, this last result becomes

$$\int_{\Gamma_1} \mathbf{F}\cdot d\mathbf{r} + \int_{\Gamma_{2-}} \mathbf{F}\cdot d\mathbf{r} = 0.$$

However, the reversal of direction of integration on path Γ_2 makes the successive paths Γ_1 and Γ_{2-} into the loop in D shown in Fig. 11.10(b). So as P and Q were any two points in D, and Γ_1 and Γ_2 were any two paths in D joining P and Q; this proves the first part of the theorem.

STEP 2 We must now prove the converse result, that if $\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = 0$ around every closed loop Γ in region D, then the field $\mathbf{F}$ is conservative in D. The proof involves reversing the argument used in Step 1. Let the arbitrary paths Γ_1 and Γ_{2-} in Fig. 11.10(b) form any loop in D, and let P and Q be any two points on the loop.

Then

$$\int_{\Gamma_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\Gamma_{2-}} \mathbf{F} \cdot d\mathbf{r} = 0,$$

but if we reverse the direction of integration along Γ_{2-}, and compensate by reversing the sign of the integral, this becomes

$$\int_{\Gamma_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\Gamma_2} \mathbf{F} \cdot d\mathbf{r}.$$

As P and Q were arbitrary points, and Γ_1 and Γ_2 are any two paths joining these points, we have succeeded in showing that the integral is path independent, so the theorem is proved. ■

Let f be a differentiable scalar function defined over a region D and let $\mathbf{F} =$ grad f be a vector field defined in terms of f. Then f is called the **potential function** for the vector field $\mathbf{F}$. The connection between potential functions and conservative fields will become clear later.

Let us now show that if a vector field $\mathbf{F}$ has a potential function f, then the function f is unique to within an arbitrary additive constant. The proof is simple. Suppose the scalar fields f and g have the same gradient in some region D, so we can write

$$\operatorname{grad}(f - g) \equiv 0.$$

Then if $\mathbf{v} \neq \mathbf{0}$ is an arbitrary vector in D, it follows from the preceding result that $\mathbf{v} \cdot \operatorname{grad}(f - g) = 0$. This shows that the directional derivative of $f - g$ is equal to zero in every direction at each point of D, and this in turn implies that $f - g =$ constant, so the result is proved.

We now establish the fundamental connection between $\mathbf{F} = \operatorname{grad} f$ and the line integral of $\mathbf{F}$ along any path Γ joining two points in a region D of space. In order to achieve this it is necessary to place some restrictions on the scalar potential function $f(x, y, z)$, the path Γ, and the region D. The function f will be assumed to have continuous first order partial derivatives in D, the path Γ in D must be continuous and piecewise smooth and comprise finitely many segments, and the region D must be open and simply connected.

The terms *open* and *simply connected* need explanation. In straightforward terms, a **simply connected** region in space can be regarded as any region that can be continuously deformed into a sphere inside of which no voids, curves, or points are missing, so it has the property that every loop in the region can be shrunk to a point that belongs to the region, without any part of the loop ever leaving the region. To understand this, consider the case of a region in space from which the points on a line are missing, and let the the loop encircle the line. Then there is no way the loop can be shrunk to a point without leaving the region, so the region is *not* simply connected (it is **multiply connected**). A region in space will be **open** if only the points on the surface of the region (its **boundary points**) are missing. A region in space is **connected** if every point in the region can be joined to every other point in the region by a piecewise continuous line that lies entirely within the region.

simply and multiply connected regions

For example, the points between two concentric spheres, the points on the surface of each of which are missing, form an *open* region that is *connected*. The region is open because its boundary points are not included in the region, and it is connected because any two points in the region can always be joined by a space curve that lies inside the region.

As another example, consider the points inside two adjacent nonintersecting spheres, each of which is connected within itself. Then the region formed by the points inside the two spheres is *not* connected, because every path joining a point in one sphere to a point in the other sphere contains points that belong to neither sphere.

THEOREM 11.4

Condition for the path independence of a line integral Let $\mathbf{F}$ be a vector field defined in an open connected region D of space, and let Γ be any path in D connecting two arbitrary points P at $\mathbf{r}_1$ and Q at $\mathbf{r}_2$ in D. Then:

(i) If the line integral $\oint_\Gamma \mathbf{F}\cdot d\mathbf{r}$ is independent of the path Γ joining $\mathbf{r}_1$ to $\mathbf{r}_2$, a scalar field f exists such that $\mathbf{F} = \operatorname{grad} f$.

(ii) If $\mathbf{F} = \operatorname{grad} f$ with $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ and $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then

$$\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = \int_P^Q (F_1\,dx + F_2\,dy + F_3\,dz) = f(Q) - f(P).$$

a condition that ensures path invariance

Proof Although not difficult, the proof of result (i) is a little harder than that of result (ii). To prove (i) it is necessary to show that if P and Q are any two points in an open connected region D, and the integral $f = \int_P^Q \mathbf{F}\cdot d\mathbf{r}$ is independent of the path Γ joining P to Q, then $\mathbf{F} = \operatorname{grad} f$.

Let P be an arbitrary point in D with coordinates (x_0, y_0, z_0), and Q be a point with coordinates (x, y_0, z_0), so that P and Q only differ in their x coordinates. By hypothesis f is independent of the path Γ from P to Q, so we can take it to be a straight line on which the general point can be written $\mathbf{r} = t\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ for $x_0 \le t \le x_1$. Let $P(x)$ be any point on Γ corresponding to $\mathbf{r} = x\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$, so $d\mathbf{r}/dt = \mathbf{i}$, and denote by $f(x)$ the integral

$$f(x) = \int_{x_0}^{x} \mathbf{F}\cdot d\mathbf{r}.$$

Then, setting $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, on path Γ we can write

$$f(x) = \int_{x_0}^{x} \mathbf{F}\cdot\left(\frac{d\mathbf{r}}{dt}\right)dt = \int_{x_0}^{x} F_1(t, y_0, z_0)dt,$$

and so

$$\begin{aligned} f(x+h) - f(x) &= \int_{x_0}^{x+h} F_1(t, y_0, z_0)dt - \int_{x_0}^{x} F_1(t, y_0, z_0)dt \\ &= \int_{x}^{x+h} \mathbf{F}_1(t, y_0, z_0)dt. \end{aligned}$$

Applying the mean value theorem for integrals (see Theorem 1.4) to the integral on the right shows that

$$f(x+h) - f(x) = hF_1(\xi, y_0, z_0),$$

where the unknown number ξ is such that $x < \xi < x + h$. The preceding expression can be rewritten in the form

$$\frac{f(x+h) - f(x)}{h} = F_1(\xi, y_0, z_0),$$

and by proceeding to the limit as $h \to 0$, when $\xi \to x$, the expression on the left reduces to $\partial f/\partial x$, because f is a function of x, y, and z, but $y = y_0$ and $z = z_0$ remain constant during the limiting process. As P was an arbitrary point in D, it follows that y_0 and z_0 are arbitrary, so we have shown that $\partial f/\partial x = F_1$. Similar arguments in which first Q is taken to be the point (x_0, y, z_0), and then to be the point (x_0, y_0, z), show that $\partial f/\partial y = F_2$ and $\partial f/\partial z = F_3$. Combining these results gives $\mathbf{F} = \operatorname{grad} f$, and the proof of (i) is complete.

To prove (ii), let the smooth path Γ joining any two points P and Q in D have the equation $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ for $a \le t \le b$. Then along Γ

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\\ &= \operatorname{grad} f \cdot \left(\frac{d\mathbf{r}}{dt}\right) = \mathbf{F}\cdot\left(\frac{d\mathbf{r}}{dt}\right),\end{aligned}$$

and so

$$\int_\Gamma \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}\cdot\left(\frac{d\mathbf{r}}{dt}\right)dt = \int_a^b \left(\frac{df}{dt}\right)dt = f(Q) - f(P),$$

and the result is proved. ■

To make effective use of Theorem 11.4 (ii) it is necessary to know when $\mathbf{F}$ is the gradient of a scalar function f. Theorem 11.5, which follows, provides both a test for a conservative field and a way of finding its associated potential function f.

THEOREM 11.5

a test for a conservative field

Testing for a conservative field and finding the potential function The vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ with components that are continuous and differentiable is a conservative field, and so is derivable from a scalar potential f, if

(i) $$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}.$$

When $\mathbf{F}$ is a conservative field the scalar potential function f is found by integrating the equations

(ii) $$\frac{\partial f}{\partial x} = F_1, \quad \frac{\partial f}{\partial y} = F_2, \quad \frac{\partial f}{\partial z} = F_3.$$

Proof If $\mathbf{F}$ is a conservative field, then a scalar potential f exists such that $\mathbf{F} = \operatorname{grad} f$, and so

$$F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Equating corresponding components gives

$$\frac{\partial f}{\partial x} = F_1, \quad \frac{\partial f}{\partial y} = F_2, \quad \frac{\partial f}{\partial z} = F_3.$$

As, by hypothesis, the components of $\mathbf{F}$ are differentiable, the equality of mixed derivatives requires that

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial F_2}{\partial x},$$

so we have established the first result in (i). The other two results are obtained in similar fashion by equating the other two mixed derivatives, so the first part of the theorem is proved. When $\mathbf{F}$ is a conservative field the scalar potential f follows by integrating the equations in (ii), and the proof of the theorem is complete. ■

EXAMPLE 11.15

Show that $\mathbf{F} = y^2z\mathbf{i} + 2xyz\mathbf{j} + (2z + xy^2)\mathbf{k}$ is a conservative field in any open connected region of space, and find the associated scalar potential f. Use the result to evaluate the line integral $I = \int_P^Q \mathbf{F} \cdot d\mathbf{r}$, where P is the point $(2, 1, 1)$ and Q is the point $(3, 2, 2)$.

Solution In the notation of Theorem 11.5 the components of $\mathbf{F}$ are $F_1 = y^2z$, $F_2 = 2xyz$, and $F_3 = 2z + xy^2$, and a routine calculation confirms that

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z},$$

in any region of space, so the $\mathbf{F}$ is a conservative field.

To find the scalar potential f we must integrate

$$\frac{\partial f}{\partial x} = y^2z, \quad \frac{\partial f}{\partial y} = 2xyz, \quad \frac{\partial f}{\partial z} = 2z + xy^2.$$

Integrating the first equation with respect to x, while regarding y and z as constants, gives

$$f = xy^2z + r(y, z),$$

where $r(y, z)$ is an arbitrary function of y and z. Combining this result with the expression for $\partial f/\partial y$ given earlier, we find that

$$\frac{\partial f}{\partial y} = 2xyz + \frac{\partial r}{\partial y} = 2xyz \quad \text{and so} \quad \frac{\partial r}{\partial y} = 0,$$

from which it follows that $r = s(z)$, with $s(z)$ an arbitrary function of z. Finally, using this result with the expression for $\partial f/\partial z$ given earlier we find that

$$\frac{\partial f}{\partial z} = xy^2 + \frac{ds}{dz} = 2z + xy^2 \quad \text{and so} \quad \frac{ds}{dz} = 2z,$$

from which it follows that $s(z) = z^2 + c$, where c is an arbitrary constant.

Combining results shows that the most general scalar potential function f associated with $\mathbf{F}$ is

$$f = xy^2z + z^2 + c.$$

As $\mathbf{F}$ is a conservative field, the line integral between any two points in an open connected region D can be evaluated using result (ii) of Theorem 11.4. However, the arbitrary constant c in f can be omitted when evaluating a line integral using the result

$$\int_P^Q \mathbf{F} \cdot d\mathbf{r} = \int_P^Q df = f(Q) - f(P),$$

because c occurs in both $f(Q)$ and $f(P)$, and so cancels. As a result, setting $f = xy^2z + z^2$ and using the notation $(xy^2z + z^2)_{(p,q,r)}$ to denote $xy^2z + z^2$ evaluated

with $x = p$, $y = q$, and $z = r$, we find that

$$I = \int_P^Q \mathbf{F} \cdot d\mathbf{r} = (xy^2z + z^2)_{(3,2,2)} - (xy^2z + z^2)_{(2,1,1)}$$
$$= 28 - 3 = 25.$$

The example that follows shows the necessity of the condition in Theorem 11.4 that the region D is *simply connected*, because if this is not the case, a line integral between two arbitrary points P and Q in D will *not* be independent of the path joining them.

EXAMPLE 11.16

Show that the two-dimensional vector field $\mathbf{F} = (\frac{-y}{x^2+y^2})\mathbf{i} + (\frac{x}{x^2+y^2})\mathbf{j}$ satisfies the conditions of Theorem 11.5 (i) in any region of space that does not contain the origin. Evaluate the integral $I = \int_\Gamma \mathbf{F} \cdot d\mathbf{r}$ when (a) Γ is the circle $x^2 + y^2 = 2$ and and (b) Γ is the square with corners P at $(1, -1)$, Q at $(3, -1)$, R at $(3, 1)$, and S at $(1, 1)$, and comment on the results.

Solution The vector $\mathbf{F}$ is indeterminate at the origin, but is defined elsewhere in the plane, where it satisfies the condition

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right).$$

This shows that $\mathbf{F}$ satisfies the two-dimensional form of Theorem 11.5 (i) in any region of the plane that does not include the origin. When the origin is excluded from the plane, vector $\mathbf{F}$ is seen to be defined in a *nonsimply connected* region.

The circle $x^2 + y^2 = 2$ and the square with its corners at $PQRS$ are shown in Fig. 11.11, from which it can be seen that the points P and S are common, so both the circle and the square represent loops in the plane containing the points P and S. The circle encloses the origin, so the points in its interior are not simply connected, while the square excludes the origin, so the points in its interior are simply connected.

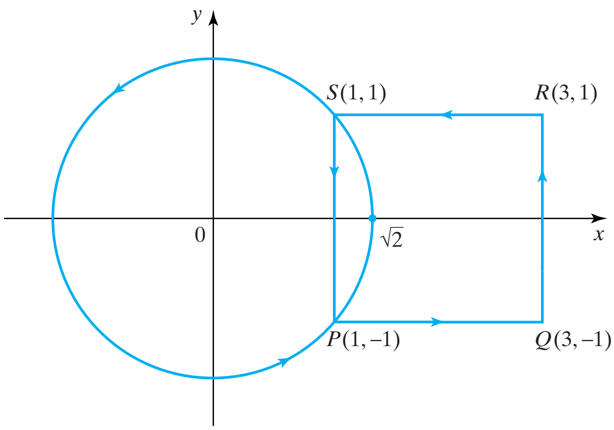


FIGURE 11.11 Two loops, each containing points P and S, in a nonsimply connected region.

Setting $x = \sqrt{2}\cos t$, $y = \sqrt{2}\sin t$ for $0 \le t \le 2\pi$ and evaluating the line integral I in case (a) gives

$$I = \int_{\Gamma}\left(\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right) = 2\pi.$$

In case (b) we have

$$\int_P^Q \mathbf{F}\cdot d\mathbf{r} = \int_1^3 \frac{dx}{x^2+1}, \quad \int_Q^R \mathbf{F}\cdot d\mathbf{r} = 3\int_{-1}^1 \frac{dy}{y^2+9}, \quad \int_R^S \mathbf{F}\cdot d\mathbf{r} = -\int_3^1 \frac{dx}{x^2+1}$$

and

$$\int_S^P \mathbf{F}\cdot d\mathbf{r} = \int_1^{-1} \frac{dy}{y^2+1}.$$

Evaluating these integrals and adding the results shows, as expected, that in case (b) the integral $I = 0$.

These results could be used to illustrate that when a region is not simply connected, the line integral between two points (in this case P and S) of a vector $\mathbf{F}$ that satisfies the conditions of Theorem 11.5 (i) will, in general, depend on the path joining the points. ■

FURTHER RESULTS

For the sake of completeness the definitions of the terms *open*, *connected*, and *simply connected* are given below in rather more detail, and they are then illustrated diagramatically by considering regions in the plane.

Definitions of open, connected, and simply connected regions

(i) A region D in space is said to be an **open** region if every point P in D can be enclosed in a sphere centered on P whose radius can always be chosen small enough that all points inside the sphere belong to D.

(ii) A region D in space is said to be **connected** if every pair of points in D can be joined by a piecewise smooth path with finitely many segments that lies entirely inside D.

(iii) A region D in space is said to be **simply connected** if every closed non-self-intersecting loop in D can be shrunk to a point in D in such a way that during the process every point on the loop remains in D.

Figure 11.12 illustrates these definitions in the case of two-dimensional regions, where a dashed boundary is used to indicate that the points on the boundary are omitted from the region. In (a), the region D is *open*, because however close P is taken to the dashed line, a circle (the two-dimensional equivalent of the sphere referred to in (i)) can always be drawn around P in such a way that all points in the circle lie in D. In (b) the region D represented by the interior of the two circles is *not* connected, because any line joining a point in one circle to a point in the other contains points that do not belong to either circle. In (c) the region D is *connected*, because any two points can always be joined by a line that lies entirely inside D.

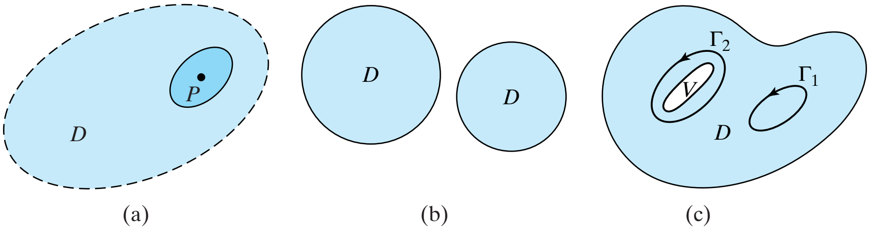


FIGURE 11.12 Regions in the plane illustrating connectivity.

However, in this case the region D is *not* simply connected, because although loop Γ_1 can be contracted to a point in such a way that every point on Γ_1 remains in D, this is not possible in the case of loop Γ_2, which encloses a void V. This last example can be visualized by considering the boundary of the void as a barrier and the loop as an elastic band. In the case of Γ_1 the elastic band can shrink to a point without hindrance, but in the case of Γ_2 this is prevented by the barrier surrounding the void.

Summary

A conservative field is one in which zero work is done when moving around a closed loop in the field and returning to the starting point. Expressed differently, a conservative field is one in which the work done when moving between two separate points is independent of the path followed between the two points. This property of conservative fields has led to this independence of a line integral on the path between two points being called the property of path invariance. The consequences of this definition have been explored and a condition has been found that ensures path invariance. A test for a conservative field has also been given.

EXERCISES 11.4

In Exercises 1 through 6 determine whether $\mathbf{F}$ is a conservative field, and if so, where.

1. $\mathbf{F} = (3x^2y^2 + yz^2)\mathbf{i} + (2x^3y + xz^2)\mathbf{j} + 2xyz\mathbf{k}$.
2. $\mathbf{F} = y\cos(xy + z^2)\mathbf{i} + x\cos(xy + z^2)\mathbf{j} + 2z\cos(xy + z^2)\mathbf{k}$.
3. $\mathbf{F} = e^x y^2\mathbf{i} + ye^x\mathbf{j} + 3xz\mathbf{k}$.
4. $\mathbf{F} = \dfrac{x}{(x^2 + y^2 + z^2)^{1/2}}\mathbf{i} - \dfrac{y}{(x^2 + y^2 + z^2)^{1/2}}\mathbf{j} + \dfrac{2z}{(x^2 + y^2 + z^2)^{1/2}}\mathbf{k}$.
5. $\mathbf{F} = \dfrac{-2xz}{(x^2 + y^2 + 2z^2)^2}\mathbf{i} + \dfrac{-2yz}{(x^2 + y^2 + 2z^2)^2}\mathbf{j} + \dfrac{x^2 + y^2 - z^2}{(x^2 + y^2 + 2z^2)^2}\mathbf{k}$.
6. $\mathbf{F} = \dfrac{z}{x^2 + y^2 + z^2}\mathbf{i} - \dfrac{y}{x^2 + y^2 + z^2}\mathbf{j} + \dfrac{x}{x^2 + y^2 + z^2}\mathbf{k}$.

In Exercises 7 to 12 show $\mathbf{F}$ is a conservative field, and by finding the scalar potential f evaluate the integral $I = \int_P^Q \mathbf{F} \cdot d\mathbf{r}$ between the given points P and Q.

7. $\mathbf{F} = (z^3 + 6xy^2)\mathbf{i} + 6x^2y\mathbf{j} + 3xz^2\mathbf{k}$ with P at $(1, 0, 1)$ and Q at $(2, 1, 0)$.
8. $\mathbf{F} = 2xz^2\cosh(x^2 + 2y^2)\mathbf{i} + 4yz^2\cosh(x^2 + 2y^2)\mathbf{j} + 2z\sinh(x^2 + 2y^2)\mathbf{k}$, with P at $(1, 1, 1)$ and Q at $(0, 2, 1)$.
9. $\mathbf{F} = e^{xyz}(1 + xyz)\mathbf{i} + x^2ze^{xyz}\mathbf{j} + x^2ye^{xyz}\mathbf{k}$, with P at $(0, 0, 0)$ and Q at $(1, 1, 2)$.
10. $\mathbf{F} = \dfrac{yz(1 - x^2)}{(1 + x^2)^2}\mathbf{i} + \dfrac{xz}{1 + x^2}\mathbf{j} + \dfrac{xy}{1 + x^2}\mathbf{k}$, with P at $(1, 1, 1)$ and Q at $(2, 2, 0)$.
11. $\mathbf{F} = 2x(1 + yz^2)\mathbf{i} + x^2z^2\mathbf{j} + 2x^2yz\mathbf{k}$, with P at $(3, 1, -1)$ and Q at $(1, 0, 2)$.
12. $\mathbf{F} = 2x(y^2 + z^2)\mathbf{i} + 2y(1 + x^2)\mathbf{j} + 2z(1 + x^2)\mathbf{k}$, with P at $(0, 1, 2)$ and Q at $(2, 0, 1)$.
13. Verify the results of Example 11.15 by performing the indicated integrations along a straight line from P to Q.

11.5 Divergence and Curl of a Vector

It is necessary to introduce two new operations involving vectors. The first operation is called the *divergence* of a vector, and it associates a *scalar function* with a differentiable vector field $\mathbf{F}$. The second operation is called the *curl* of a vector, and it associates a *vector function* with the vector $\mathbf{F}$. If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a differentiable vector field, the **divergence** of $\mathbf{F}$, written div $\mathbf{F}$, is the scalar function defined in terms of cartesian coordinates as

divergence of a vector

$$\operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \tag{39}$$

The divergence of the vector $\mathbf{F}$ can also be expressed in terms of the operator "del" defined in (25) as

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z},$$

by writing

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}), \tag{40}$$

where the mutual orthogonality of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ coupled with the fact that they are constant vectors causes the expression on the right of (40) to be reduced to the expression on the right of (39), with the operation $\nabla \cdot \mathbf{F}$ being read "del dot $\mathbf{F}$." The form taken by div $\mathbf{F}$ in more general coordinate systems is derived in Section 11.6.

At this stage, for simplicity, the definition of div $\mathbf{F}$ is expressed in terms of cartesian coordinates, though it will be shown later that div $\mathbf{F}$ is, in fact, independent of any coordinate system. In the next chapter it will be shown that div $\mathbf{F}$ can be interpreted as the flux of the normal component of the vector $\mathbf{F}$ that crosses the surface of a unit volume in a unit time. This means that when div $\mathbf{F}$ is positive, there is a net *flow* of $\mathbf{F}$ *out* of the volume, and when div $\mathbf{F}$ is negative, there is a net *flow* of $\mathbf{F}$ *into* the volume.

In anticipation of the next chapter, we give a heuristic derivation of div $\mathbf{F}$ in terms of cartesian coordinates that shows how div $\mathbf{F}$ can be defined differently, and at the same time illustrates its physical significance. Consider the small cube of side a shown in Fig. 11.13 with faces normal to the coordinate axes, and take the positive direction of the normal to each face of the cube to be the one directed *out* of the cube. The normal component of $\mathbf{F}$ *entering* face A is $F_2(x, y_0, z)$, and the normal component of $\mathbf{F}$ *leaving* face B is $F_2(x, y_0 + a, z)$, where from Taylor's theorem for functions of several variables, to first order in a we have $F_2(x, y_0 + a, z) = F_2(x, y_0, z) + a\partial F_2(x, y_0, z)/\partial y$.

Consequently, if we average $F_2(x, y_0, z)$ over face A and denote the result by $\tilde{F}_2$, the integral of $F_2(x, y_0, z)$ over face A is approximately equal to $a^2\tilde{F}_2$, while the integral over face B is approximately equal to $a^2[\tilde{F}_2 + a\partial\tilde{F}_2/\partial y]$, so the change of the flux of $\mathbf{F}$ from face A to face B is approximately $a^3\partial\tilde{F}_2/\partial y$. Similar results apply to the other pairs of faces, so denoting the surface of the cube by S, and letting F_n

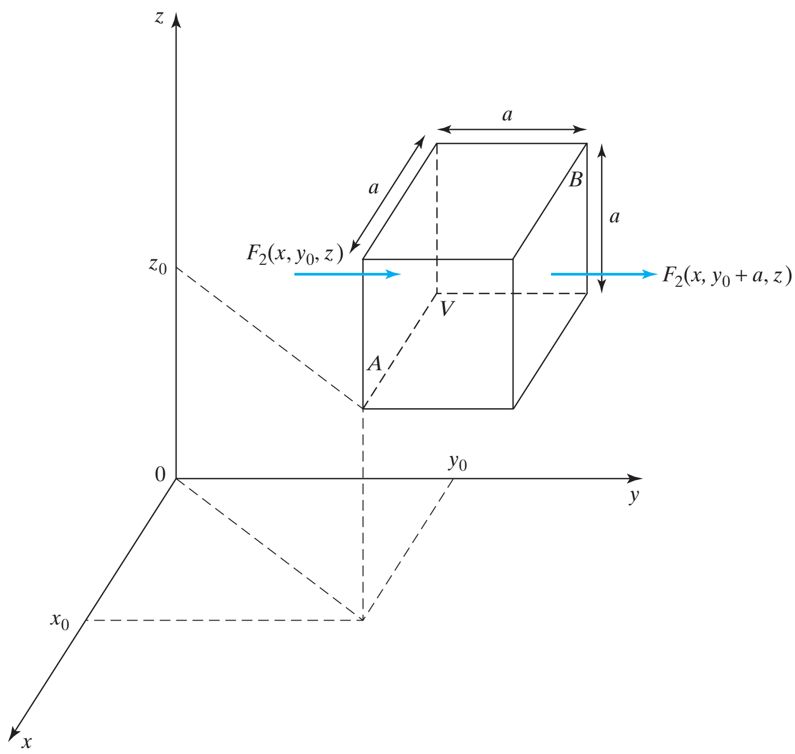


FIGURE 11.13 A representative cubic element.

denote the component of **F** normal to S, positive when outward, with dS a surface element of area of a face, we have

$$\lim_{a\to 0}\frac{1}{a^3}\iint_S F_n\,dS = \lim_{a\to 0}\frac{1}{a^3}\left(a^3\frac{\partial \tilde{F}_1}{\partial x} + a^3\frac{\partial \tilde{F}_2}{\partial y} + a^3\frac{\partial \tilde{F}_3}{\partial z}\right) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

The expression on the right is div **F**, so this result shows that the divergence of a vector field **F** in cartesian coordinates is the limit of the flux of the normal component of **F** through the surface S bounding a volume as the volume tends to zero. A different form of argument used in the next chapter will show that for *any* volume V with surface S and element of surface area dS, independently of any coordinate system

a different interpretation of div F

$$\text{div}\,\mathbf{F} = \lim_{V\to 0}\frac{1}{V}\iint_S F_n\,dS.$$

It is helpful to interpret this result in terms of the flow of a liquid. If we identify **q** with the liquid velocity vector, V with the volume occupied by the liquid, and S with the surface enclosing V, the product $q_n\,dS$, with q_n the component of **q** normal to dS, is seen to be the volume of liquid crossing the surface element dS in a unit time. Consequently, $\iint_S F_n\,dS$ is the total volume of liquid leaving through the surface S in a unit time. As a liquid can be considered to be incompressible, provided the volume contains neither a *source* of liquid (a point in V through which liquid enters) nor a *sink* (a point in V through which liquid is extracted), it follows that $\iint_S F_n\,dS$ will be zero for an incompressible fluid.

Thus, in an incompressible liquid free from sources and sinks, div $\mathbf{q} = 0$. If sources and sinks occur in the liquid, their strengths can be found by enclosing each in a small volume and then letting it become arbitrarily small, in which case a *positive* value of div $\mathbf{q}$ will correspond to a source and a *negative* value to a sink.

If, instead of a liquid, the flow of a gas is involved, the compressibility of a gas causes its density to vary from point to point, so then, in general, the value of div $\mathbf{q}$ will depend on position and, if the flow is unsteady, also on the time.

EXAMPLE 11.17

Find div $\mathbf{F}$ when $\mathbf{F} = xy^2\mathbf{i} + 3yz\mathbf{j} - 4xz\mathbf{k}$.

Solution From (39) div $\mathbf{F} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(3yz) + \frac{\partial}{\partial z}(-4xz) = y^2 + 3z - 4x$. ■

We have seen that provided f is suitably differentiable, grad f is a vector, so when f is twice differentiable it is appropriate to examine the operation div (grad f). This is usually written div grad f, because no ambiguity arises when the brackets are omitted. By definition

$$\begin{aligned}\text{div grad } f &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot \left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}\right) \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f, \end{aligned} \tag{41}$$

and so div grad $f = \Delta f$ is simply the **Laplacian** of f.

THEOREM 11.6

fundamental properties of the divergence operator

Properties of the divergence operator Let the vector fields $\mathbf{F}$ and $\mathbf{G}$ and the scalar fields ϕ and ψ be a suitably differentiable, and let a and b be constants. Then the divergence operator has the following properties:

(i) $\text{div}(a\mathbf{F}) = a\,\text{div}\,\mathbf{F}$

(ii) $\text{div}(a\mathbf{F} + b\mathbf{G}) = a\,\text{div}\,\mathbf{F} + b\,\text{div}\,\mathbf{G}$

(iii) $\text{div}(\phi\mathbf{F}) = \phi\,\text{div}\,\mathbf{F} + \mathbf{F}\cdot\nabla\phi$

(iv) $\text{div}(\text{grad}\,\phi) = \Delta\phi$

(v) $\text{div}(\phi\nabla\psi) = \phi\Delta\psi + \text{grad}\,\phi \cdot \text{grad}\,\psi = \phi\Delta\psi + \nabla\phi\cdot\nabla\psi$

(vi) $\text{div}(\phi\nabla\psi) - \text{div}(\psi\nabla\phi) = \phi\Delta\psi - \psi\Delta\phi$

Proof The derivation of these results follows directly from the definition of the divergence of a vector in (39). So, as (iv) has already been established, we will only prove (iii) and leave the other results as exercises.

If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, it follows that $\phi\mathbf{F} = \phi F_1\mathbf{i} + \phi F_2\mathbf{j} + \phi F_3\mathbf{k}$, and so

$$\begin{aligned}\text{div}(\phi\mathbf{F}) &= \frac{\partial}{\partial x}(\phi F_1) + \frac{\partial}{\partial y}(\phi F_2) + \frac{\partial}{\partial z}(\phi F_3) \\ &= \phi\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) + F_1\frac{\partial\phi}{\partial x} + F_2\frac{\partial\phi}{\partial y} + F_3\frac{\partial\phi}{\partial z} \\ &= \phi\,\text{div}\,\mathbf{F} + \mathbf{F}\cdot\nabla\phi.\end{aligned}$$

■

the definition of curl F

When expressed in terms of cartesian coordinates, the **curl** of the vector $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is defined as

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}. \tag{42}$$

This form of the definition of curl **F** is more easily remembered when expressed symbolically as the determinant

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}, \tag{43}$$

or in terms of the operator "del" as

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}), \tag{44}$$

where it is to be understood that the differentiations are to be performed before finding the cross products, and the operation $\nabla \times \mathbf{F}$ is read as "del cross **F**."

EXAMPLE 11.18

Find curl **F** given that $\mathbf{F} = xy\mathbf{i} + z\mathbf{j} + yz\mathbf{k}$.

Solution Using (43) we have

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & z & yz \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(z)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial z}(xy)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial y}(xy)\right)\mathbf{k} \\ &= (z-1)\mathbf{i} - x\mathbf{k}.\end{aligned}$$

■

EXAMPLE 11.19

Show that if ϕ is any scalar function with continuous first and second order derivatives, then $\operatorname{curl}(\operatorname{grad}\phi) \equiv \mathbf{0}$.

Solution By definition $\operatorname{grad}\phi = \phi_x\mathbf{i} + \phi_y\mathbf{j} + \phi_z\mathbf{k}$, so from (44)

$$\operatorname{curl}(\operatorname{grad}\phi) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (\phi_x\mathbf{i} + \phi_y\mathbf{j} + \phi_z\mathbf{k}).$$

After we use the properties of the vector product with the mutually orthogonal unit vectors **i**, **j**, and **k**, this reduces to

$$\operatorname{curl}(\operatorname{grad}\phi) = \frac{\partial}{\partial x}(\phi_y)\mathbf{k} - \frac{\partial}{\partial x}(\phi_z)\mathbf{j} - \frac{\partial}{\partial y}(\phi_x)\mathbf{k} + \frac{\partial}{\partial y}(\phi_z)\mathbf{i} + \frac{\partial}{\partial z}(\phi_x)\mathbf{j} - \frac{\partial}{\partial z}(\phi_y)\mathbf{i}.$$

By hypothesis ϕ has continuous partial derivatives up to and including order 2, so there is equality of mixed derivatives. As a result $\phi_{xy} = \phi_{yx}$, showing that the **k** component of $\operatorname{curl}(\operatorname{grad}\phi)$ vanishes. The **j** and **i** components of $\operatorname{curl}(\operatorname{grad}\phi)$ vanish for the same reason so that $\operatorname{curl}(\operatorname{grad}\phi) \equiv \mathbf{0}$. ■

The operators grad, div, and curl can be combined in various ways that lead to identities, the results of which are listed in the next theorem. These identities are useful when manipulating vector operations. In some of the entries the notation $(\mathbf{F}\cdot\nabla)\mathbf{G}$ is used, and if $\mathbf{F}=F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}$ and $\mathbf{G}=G_1\mathbf{i}+G_2\mathbf{j}+G_3\mathbf{k}$ this is to be interpreted as the vector

$$
\begin{aligned}
(\mathbf{F}\cdot\nabla)\mathbf{G} &= \left[(F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k})\cdot\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\right](G_1\mathbf{i}+G_2\mathbf{j}+G_3\mathbf{k})\\
&= \left(F_1\frac{\partial}{\partial x}+F_2\frac{\partial}{\partial y}+F_3\frac{\partial}{\partial z}\right)(G_1\mathbf{i}+G_2\mathbf{j}+G_3\mathbf{k}).
\end{aligned}
$$

THEOREM 11.7

combining grad, div, and curl

Properties of combinations of grad, div, and curl Let $\mathbf{F}$ and $\mathbf{G}$ be vector functions and let ϕ be a scalar function, all of which are suitably differentiable. Then the following identities hold.

(i) $\operatorname{curl}(\operatorname{grad}\phi)=\mathbf{0}$

(ii) $\operatorname{div}(\operatorname{curl}\mathbf{F})=0$

(iii) $\operatorname{curl}(\phi\mathbf{F})=\phi\operatorname{curl}\mathbf{F}-\mathbf{F}\times\operatorname{grad}\phi$

(iv) $\operatorname{grad}(\mathbf{F}\cdot\mathbf{G})=\mathbf{F}\times\operatorname{curl}\mathbf{G}+\mathbf{G}\times\operatorname{curl}\mathbf{F}+(\mathbf{F}\cdot\nabla)\mathbf{G}+(\mathbf{G}\cdot\nabla)\mathbf{F}$

(v) $\operatorname{div}(\mathbf{F}\times\mathbf{G})=\mathbf{G}\cdot\operatorname{curl}\mathbf{F}-\mathbf{F}\cdot\operatorname{curl}\mathbf{G}$

(vi) $\operatorname{curl}(\mathbf{F}\times\mathbf{G})=\mathbf{F}\operatorname{div}\mathbf{G}-\mathbf{G}\operatorname{div}\mathbf{F}+(\mathbf{G}\cdot\nabla)\mathbf{F}-(\mathbf{F}\cdot\nabla)\mathbf{G}$

(vii) $\operatorname{curl}(\operatorname{curl}\mathbf{F})=\operatorname{grad}(\operatorname{div}\mathbf{F})-\Delta\mathbf{F}$

Proof Result (i) has already been established. As the other results follow in similar fashion from the definitions of the gradient, divergence, and curl operators, the remaining proofs are left as exercises. ■

The expression for curl $\mathbf{F}$ in more general coordinate systems is derived in Section 11.6, but a different definition of curl $\mathbf{F}$ together with a physical interpretation will be postponed until after the discusion of Stokes' theorem in the next chapter.

Theorem 11.7 provides a test for conservative vector fields $\mathbf{F}$. Although the test is equivalent to the test in Theorem 11.5 (i), it is in a more easily remembered form. By definition, a vector field $\mathbf{F}$ is a *conservative field* if $\mathbf{F}=\operatorname{grad} f$, but from (i) of Theorem 11.7, if $\mathbf{F}=\operatorname{grad} f$ then $\operatorname{curl}\mathbf{F}=\mathbf{0}$, and it is this last result that provides the test. However, if after establishing that $\mathbf{F}$ is a conservative field its associated potential function f is required, it must be found by integrating the equations in Theorem 11.5 (ii), as illustrated in Example 11.14.

using curl F to test for a conservative field

Curl test for a conservative vector field

A vector field $\mathbf{F}$ is conservative, that is, it is $\mathbf{F}=\operatorname{grad} f$ where f is the associated scalar potential, if $\operatorname{curl}\mathbf{F}=\mathbf{0}$.

EXAMPLE 11.20

For what values of a and b is the vector field $\mathbf{F}=(x+z)\mathbf{i}+a(y+z)\mathbf{j}+b(x+y)\mathbf{k}$ a conservative field?

Solution

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+z & a(y+z) & b(x+y) \end{vmatrix} = (b-a)\mathbf{i} + (1-b)\mathbf{j},$$

so curl $\mathbf{F} = \mathbf{0}$ if $b - a = 0$ and $1 - b = 0$. Consequently, $\mathbf{F}$ will be a conservative field if $a = b = 1$. ■

EXAMPLE 11.21 Find curl(curl $\mathbf{F}$) given that $\mathbf{F} = x^2y^2\mathbf{i} + y^2z^2\mathbf{j} + x^2z^2\mathbf{k}$.

Solution To calculate curl(curl $\mathbf{F}$), we will use result (vii) of Theorem 11.7. We have

$$\operatorname{div}\mathbf{F} = 2xy^2 + 2yz^2 + 2zx^2,$$

so

$$\operatorname{grad}(\operatorname{div}\mathbf{F}) = (2y^2 + 4xz)\mathbf{i} + (2z^2 + 4xy)\mathbf{j} + (2x^2 + 4yz)\mathbf{k}.$$

Next,

$$\begin{aligned}\Delta\mathbf{F} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(x^2y^2\mathbf{i} + y^2z^2\mathbf{j} + x^2z^2\mathbf{k}) \\ &= 2(x^2+y^2)\mathbf{i} + 2(y^2+z^2)\mathbf{j} + 2(x^2+z^2)\mathbf{k},\end{aligned}$$

so combining results gives

$$\operatorname{curl}(\operatorname{curl}\mathbf{F}) = (4xz - 2x^2)\mathbf{i} + (4xy - 2y^2)\mathbf{j} + (4yz - 2z^2)\mathbf{k}.$$ ■

Vector fields, line integrals, the theory, application, and evaluation of multiple integrals, and the vector operators grad, div, and curl are all defined and their properties developed in standard calculus and analytic geometry texts such as those in references [1.1], [1.2], [1.5], [1.6], and [1.7]. Reference [5.6] gives a concise summary of these results together with numerous examples. More advanced and detailed accounts, where the emphasis is placed on a vector treatment, are to be found in references [5.1], [5.2], and [1.4].

Summary

The previous section introduced the gradient operator, where it was shown that it acts on a scalar function of position to produce a vector. The present section introduced two more vector operators called the divergence and curl operators. The divergence operator was seen to act on a vector to produce a scalar, while the curl operator acted on a vector to produce another vector. The general operational properties of the divergence and curl operators were developed together with the results of combining all three vector operators.

EXERCISES 11.5

In Exercises 1 through 4, find div $\mathbf{F}$ for the given vector function $\mathbf{F}$.

1. $\mathbf{F} = x^2y\mathbf{i} + y^2z^2\mathbf{j} + xz^3\mathbf{k}$.
2. $\mathbf{F} = (1-x^2)\mathbf{i} + \sin yz\mathbf{j} + e^{xyz}\mathbf{k}$.
3. $\mathbf{F} = 3x^2\mathbf{i} + 2x^2y^2\mathbf{j} + x\mathbf{k}$.
4. $\mathbf{F} = \cos x\mathbf{i} + \sin y\mathbf{j} + z^2\mathbf{k}$.
5. Prove that $\operatorname{div}(\phi\mathbf{F}) = \phi \operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla\phi$ (Theorem 11.6 (iii)).
6. Prove that $\operatorname{div}(\phi\nabla\psi) = \phi\Delta\psi + \nabla\phi\cdot\nabla\psi$ (Theorem 11.6 (v)).

In Exercises 7 through 10 find curl $\mathbf{F}$ for the given vector function $\mathbf{F}$.

7. $\mathbf{F} = xyz^2\mathbf{i} + x^2yz\mathbf{j} + xy^2\mathbf{k}$.

8. $\mathbf{F} = \sinh xy\mathbf{i} + \cosh yz\mathbf{j} + xyz\mathbf{k}$.

9. $\mathbf{F} = \arctan \frac{x}{y}\mathbf{i} + \ln(x^2 + 2y^2)^{1/2}\mathbf{j} + y\mathbf{k}$.

10. $\mathbf{F} = (x^2 + y^2 + z^2)^{1/2}\mathbf{i} + (x^2 + y^2 + z^2)^{1/2}\mathbf{j} + x\mathbf{k}$.

11. Prove that $\operatorname{div}(\operatorname{curl} \mathbf{F}) \equiv 0$ (Theorem 11.7 (ii)).

12. Prove that $\operatorname{curl}(\phi\mathbf{F}) \equiv \phi \operatorname{curl} \mathbf{F} - \mathbf{F} \times \operatorname{grad} \phi$ (Theorem 11.7 (iii)).

13. Prove that $\operatorname{grad}(\mathbf{F} \cdot \mathbf{G}) \equiv \mathbf{F} \times \operatorname{curl} \mathbf{G} + \mathbf{G} \times \operatorname{curl} \mathbf{F} + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F}$ (Theorem 11.7 (iv)).

14. Prove that $\operatorname{div}(\mathbf{F} \times \mathbf{G}) \equiv \mathbf{G} \cdot \operatorname{curl} \mathbf{F} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G}$ (Theorem 11.7 (v)).

15. Prove that $\operatorname{curl}(\mathbf{F} \times \mathbf{G}) \equiv \mathbf{F} \operatorname{div} \mathbf{G} - \mathbf{G} \operatorname{div} \mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$ (Theorem 11.7 (vi)).

16. Prove that $\operatorname{curl}(\operatorname{curl} \mathbf{F}) = \operatorname{grad}(\operatorname{div} \mathbf{F}) - \Delta\mathbf{F}$ (Theorem 11.7 (vii)).

17. Find $\operatorname{curl}(\operatorname{curl} \mathbf{F})$ given that $\mathbf{F} = 3xyz\mathbf{i} + 2y\mathbf{j} - 4z\mathbf{k}$.

In Exercises 17 and 20 use the curl test to see if or where the vector field $\mathbf{F}$ is conservative.

18. $\mathbf{F} = yz\cosh(xyz + y^2)\mathbf{i} + (xz + 2y)\cosh(xyz + y^2)\mathbf{j} + 2xy\cosh(xyz + y^2)\mathbf{k}$.

19. $\mathbf{F} = 2xy^2\mathbf{i} + (2x^2y + 6yz^3)\mathbf{j} + 9y^2z^2\mathbf{k}$.

20. $\mathbf{F} = \dfrac{1}{(x^2 + y^2 + z^2)^{1/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$.

21. $\mathbf{F} = \dfrac{1}{(1 + x^2 + 2y^2z)}(2x\mathbf{i} + 4yz\mathbf{j} + 2y^2\mathbf{k})$.

11.6 Orthogonal Curvilinear Coordinates

The geometrical configuration of a physical problem often suggests the most appropriate coordinate system that should be used when seeking its solution. For example, heat conduction in a cylindrical rod suggests the use of cylindrical polar coordinates with the z-axis aligned with the axis of the rod, whereas the distribution of an electric field inside a spherical cavity suggests the use of spherical polar coordinates. When problems of this nature are expressed in terms of vectors, and the operators grad, div, and curl are involved, it becomes necessary to find the form taken by these operators in different systems of curvilinear coordinates. The reader who wishes to omit the derivation of the main results of this section should proceed directly to Theorem 11.8 after studying the definition of an orthogonal system of curvilinear coordinates and the meaning of the scale factors h_1, h_2, and h_3.

In what follows, in order to unify notation, it is convenient to denote the usual cartesian coordinates x, y, and z by x_1, x_2, and x_3 and a general system of curvilinear coordinates by q_1, q_2, and q_3, where the two systems are related by the equations

$$x_1 = x_1(q_1, q_2, q_3), \quad x_2 = x_2(q_1, q_2, q_3), \quad x_3 = x_3(q_1, q_2, q_3). \tag{45}$$

For the curvilinear coordinates q_1, q_2, and q_3 to be equivalent to the cartesian coordinate system x_1, x_2, and x_3 it is necessary that equations (45) can be solved uniquely in the form

$$q_1 = q_1(x_1, x_2, x_3), \quad q_2 = q_2(x_1, x_2, x_3), \quad q_3 = q_3(x_1, x_2, x_3), \tag{46}$$

so that one point in cartesian coordinates corresponds to only one point in curvilinear coordinates, and conversely. As derivatives of functions occur in grad, div, and curl, it is necessary that the coordinate functions x_1, x_2, and x_3, as functions of q_1, q_2, and q_3 in (45), are all suitably differentiable with respect to their arguments.

Taking the total differentials of the coordinate transformations in (45), we have

$$dx_1 = \frac{\partial x_1}{\partial q_1}dq_1 + \frac{\partial x_1}{\partial q_2}dq_2 + \frac{\partial x_1}{\partial q_3}dq_3, \quad dx_2 = \frac{\partial x_2}{\partial q_1}dq_1 + \frac{\partial x_2}{\partial q_2}dq_2 + \frac{\partial x_2}{\partial q_3}dq_3$$

$$dx_3 = \frac{\partial x_3}{\partial q_1}dq_1 + \frac{\partial x_3}{\partial q_2}dq_2 + \frac{\partial x_3}{\partial q_3}dq_3. \tag{47}$$

These results can be written in the matrix form

$$d\mathbf{x} = \mathbf{J}\, d\mathbf{q}, \tag{48}$$

where

$$d\mathbf{x} = \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}, \quad d\mathbf{q} = \begin{bmatrix} dq_1 \\ dq_2 \\ dq_3 \end{bmatrix}, \quad \text{and} \quad \mathbf{J} = \begin{bmatrix} \dfrac{\partial x_1}{\partial q_1} & \dfrac{\partial x_1}{\partial q_2} & \dfrac{\partial x_1}{\partial q_3} \\ \dfrac{\partial x_2}{\partial q_1} & \dfrac{\partial x_2}{\partial q_2} & \dfrac{\partial x_2}{\partial q_3} \\ \dfrac{\partial x_3}{\partial q_1} & \dfrac{\partial x_3}{\partial q_2} & \dfrac{\partial x_3}{\partial q_3} \end{bmatrix}. \tag{49}$$

The matrix vector linear differential elements $d\mathbf{x}$ and $d\mathbf{q}$ will be uniquely related by (48) provided matrix $\mathbf{J}$ is nonsingular, so the coordinate transformations (45) must be such that $J = \det \mathbf{J} \neq 0$, where

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial q_1} & \dfrac{\partial x_2}{\partial q_1} & \dfrac{\partial x_3}{\partial q_1} \\ \dfrac{\partial x_1}{\partial q_2} & \dfrac{\partial x_2}{\partial q_2} & \dfrac{\partial x_3}{\partial q_2} \\ \dfrac{\partial x_1}{\partial q_3} & \dfrac{\partial x_2}{\partial q_3} & \dfrac{\partial x_3}{\partial q_3} \end{vmatrix}. \tag{50}$$

the Jacobian of a transformation

The determinant J is called the **Jacobian** of the transformation, and it will be shown later that the absolute value of the Jacobian occurs as a scale factor in the **volume element** in orthogonal curvilinear coordinates. Thus, the vanishing of the Jacobian signifying nonuniqueness in the transformations (45) and (46) also corresponds to the failure of the curvilinear coordinate system to define a volume element.

Carl Gustav Jacobi (1804–1851)
A German mathematician who studied at the University of Berlin and obtained his doctorate in 1825. In 1827 he was appointed Extraordinary Professor of Mathematics at Königsberg and, after two years, he was promoted to Ordinary Professor of Mathematics. In 1842 he moved to Berlin where he remained until his death. His most important work was in connection with elliptic functions, but he also made important contributions to number theory, ordinary and partial differential equations, and the calculus of variations. He was an outstanding teacher of mathematics.

Keeping q_1 and $q_1 + dq_1$ constant defines two curvilinear surfaces in space, and four further curvilinear surfaces are defined by keeping q_2 and $q_2 + dq_2$ constant, and q_3 and $q_3 + dq_3$ constant. Taken together, the region between these six curvilinear surfaces defines the volume element dV in space shown in Fig. 11.14.

general and orthogonal curvilinear coordinates

Allowing q_1 to vary while holding q_2 and q_3 constant in (45) will generate a *curvilinear coordinate line* in space along which only q_1 changes. Similarly, allowing q_2 to vary while holding q_1 and q_3 constant, and then q_3 to vary while holding q_1 and q_2 constant, will generate curvilinear coordinate lines in space along which, respectively, only q_2 and q_3 vary. If a general point A in space shown in Fig. 11.14 is considered, there will be three curvilinear coordinate lines passing through the point. A curvilinear coordinate system will be said to be an **orthogonal** system if at every point in space the three tangents to the coordinate lines at their point of intersection

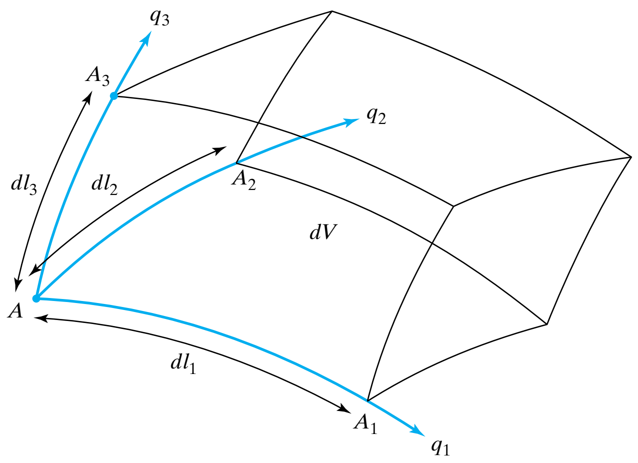


FIGURE 11.14 The curvilinear volume element dV.

are mutually orthogonal (perpendicular). Such coordinate systems are also considered to be *orthogonal* if the orthogonality condition fails at a single point or along a line. In what follows, only orthogonal coordinate systems will be considered.

the volume element

With the linear differential length elements $AA_1 = dl_1$, $AA_2 = dl_2$, and $AA_3 = dl_3$, the orthogonality of the curvilinear coordinate system implies that in terms of curvilinear coordinates the linear volume element dV in Fig. 11.14 is given by

$$dV = dl_1\, dl_2\, dl_3. \tag{51}$$

Now, in Fig. 11.14, let A be the point (x_1, x_2, x_3) and A_1 be the point $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$, where dx_1, dx_2, and dx_3 are the linear differential elements in cartesian coordinates. To find the linear differential length element dl_1 from A to A_1, we apply the Pythagoras theorem to the mutually orthogonal linear differential length elements dx_1, dx_2, and dx_3, when we obtain

$$dl_1^2 = dx_1^2 + dx_2^2 + dx_3^2, \tag{52}$$

However along AA_1 only q_1 varies, so as

$$dx_1 = \frac{\partial x_1}{\partial q_1} dq_1, \quad dx_2 = \frac{\partial x_2}{\partial q_1} dq_1, \quad dx_3 = \frac{\partial x_3}{\partial q_1} dq_1, \tag{53}$$

the square of the linear differential length element in (52) becomes

$$dl_1^2 = \left[\left(\frac{\partial x_1}{\partial q_1} \right)^2 + \left(\frac{\partial x_2}{\partial q_1} \right)^2 + \left(\frac{\partial x_3}{\partial q_1} \right)^2 \right] dq_1^2. \tag{54}$$

Similar arguments show that if dl_2 and dl_3 are the linear differential length elements along AA_2 and AA_3, then

$$dl_2^2 = \left[\left(\frac{\partial x_1}{\partial q_2} \right)^2 + \left(\frac{\partial x_2}{\partial q_2} \right)^2 + \left(\frac{\partial x_3}{\partial q_2} \right)^2 \right] dq_2^2, \tag{55}$$

and

$$dl_3^2 = \left[\left(\frac{\partial x_1}{\partial q_3}\right)^2 + \left(\frac{\partial x_2}{\partial q_3}\right)^2 + \left(\frac{\partial x_3}{\partial q_3}\right)^2\right] dq_3^2. \tag{56}$$

the scale factors h_1, h_2, h_3

We now adopt the standard notation and define the **scale factors** h_1, h_2, and h_3, with respect to the coordinates q_1, q_2, and q_3 in transformations (45), by

$$h_1 = \left[\left(\frac{\partial x_1}{\partial q_1}\right)^2 + \left(\frac{\partial x_2}{\partial q_1}\right)^2 + \left(\frac{\partial x_3}{\partial q_1}\right)^2\right]^{1/2} \tag{57}$$

$$h_2 = \left[\left(\frac{\partial x_1}{\partial q_2}\right)^2 + \left(\frac{\partial x_2}{\partial q_2}\right)^2 + \left(\frac{\partial x_3}{\partial q_2}\right)^2\right]^{1/2} \tag{58}$$

$$h_3 = \left[\left(\frac{\partial x_1}{\partial q_3}\right)^2 + \left(\frac{\partial x_2}{\partial q_3}\right)^2 + \left(\frac{\partial x_3}{\partial q_3}\right)^2\right]^{1/2}. \tag{59}$$

In terms of h_1, h_2, and h_3 the linear differential line elements dl_1, dl_2, and dl_3 in rectangular curvilinear coordinates defined in (54) to (56) become

$$dl_1 = h_1 dq_1, \quad dl_2 = h_2 dq_2, \quad dl_3 = h_3 dl_3. \tag{60}$$

If the general linear differential length element from A to B in Fig. 11.14 is denoted by ds, then as the coordinate system is orthogonal,

$$ds^2 = dl_1^2 + dl_2^2 + dl_3^2, \tag{61}$$

so it follows from (60) that

$$ds^2 = h_1^2 dq_1^2 + h_2^2\, dq_2^2 + h_3^2\, dq_3^2. \tag{62}$$

In terms of the scale factors the linear differential volume element dV in (51) becomes

$$dV = h_1 h_2 h_3\, dq_1\, dq_2\, dq_3. \tag{63}$$

It can be seen from this last result that the coordinate transformations (45) will fail to define a volume element in curvilinear coordinates if a scale factor vanishes. From the definitions of the scale factors, this can only happen if all of the partial derivatives in a scale factor vanish, but when this occurs the Jacobian determinant J will have a zero row, and so will also vanish. This is to be expected, because it is known from calculus that when the Jacobian vanishes, the transformation between the coordinate systems ceases to be one to one.

To understand the geometrical interpretation of the Jacobian, we make use of the elementary result from vector analysis that the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ can be interpreted as the volume of the parallelepiped with sides given by vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ that meet at a point. The value of this scalar triple product is equal to the determinant with the elements of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ as its first, second, and third rows,

respectively. Considering dx_1, dx_2, and dx_3 in (47) as vectors in the curvilinear coordinate system, we see that the linear differential volume element $dV = dx_1 dx_2 dx_3$ can be written

$$\pm dV = \begin{vmatrix} \frac{\partial x_1}{\partial q_1}dq_1 & \frac{\partial x_2}{\partial q_1}dq_1 & \frac{\partial x_3}{\partial q_1}dq_1 \\ \frac{\partial x_1}{\partial q_2}dq_2 & \frac{\partial x_2}{\partial q_2}dq_2 & \frac{\partial x_3}{\partial q_2}dq_2 \\ \frac{\partial x_1}{\partial q_3}dq_3 & \frac{\partial x_2}{\partial q_3}dq_3 & \frac{\partial x_3}{\partial q_3}dq_3 \end{vmatrix} = \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_2}{\partial q_1} & \frac{\partial x_3}{\partial q_1} \\ \frac{\partial x_1}{\partial q_2} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_3}{\partial q_2} \\ \frac{\partial x_1}{\partial q_3} & \frac{\partial x_2}{\partial q_3} & \frac{\partial x_3}{\partial q_3} \end{vmatrix} dq_1\, dq_2\, dq_3. \tag{64}$$

the Jacobian and the volume element

As a volume element is essentially nonnegative, this can be expressed in terms of the Jacobian J of the transformation as

$$dV = \pm J\, dq_1\, dq_2\, dq_3, \tag{65}$$

where the sign in (65) is chosen to make the expression on the right positive. A comparison of (63) and (65) then shows that the absolute value of the Jacobian J is equal to the product of the scale factors forming the scale factor for the linear volume element dV, and so

$$h_1 h_2 h_3 = \pm J, \tag{66}$$

where the sign is chosen to make the expression on the right positive.

EXAMPLE 11.22

Find the scale factors, the linear differential length elements along the curvilinear coordinate lines, the square of the general linear differential length element ds, the linear differential volume element dV, and the Jacobian for (a) cylindrical polar coordinates and (b) spherical polar coordinates.

Solution

(a) In cylindrical polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, so to relate this system to the general one just considered, we must make the identifications $x_1 = x$, $x_2 = y$, $x_3 = z$, $q_1 = r$, $q_2 = \theta$, and $q_3 = z$. When this is done, substitution into (57) to (59) shows that

$$h_1 = 1, \quad h_2 = r, \quad h_3 = 1,$$

so from (60) the linear differential length elements along the curvilinear coordinate lines are

$$dl_1 = dr, \quad dl_2 = r\, d\theta, \quad dl_3 = dz.$$

It then follows from (62) that the square of the general linear differential length element ds is

$$ds^2 = dr^2 + r^2\, d\theta^2 + dz^2,$$

and from (63) that the linear differential volume element in terms of cylindrical polar coordinates is

$$dV = r\, dr\, d\theta\, dz.$$

The Jacobian of the transformation

$$J = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r,$$

in agreement with (66).

The transformation ceases to be one to one when $r = 0$, because then $h_2 = 0$, though this is to be expected because $r = 0$ is the z-axis along which θ is indeterminate.

(b) In spherical polar coordinates $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, so to relate this system to the general one just considered we must make the identifications $x_1 = x$, $x_2 = y$, $x_3 = z$, $q_1 = r$, $q_2 = \phi$, and $q_3 = \theta$. When this is done, substitution into (57) to (59) shows that

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta,$$

so from (60) the linear differential length elements along the curvilinear coordinate lines are

$$dl_1 = dr, \quad dl_2 = r\,d\phi, \quad dl_3 = r\sin\theta\,d\theta$$

As in (a), it follows from (62) that the square of the general linear differential length element ds is

$$ds^2 = dr^2 + r^2\sin^2\theta\,d\theta^2 + r^2\,d\phi^2$$

and from (63) that the linear differential volume element in terms of spherical polar coordinates is

$$dV = r^2\sin\theta\,dr\,d\theta\,d\phi.$$

The Jacobian of the transformation

$$J = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \end{vmatrix} = -r^2\sin\theta,$$

and in agreement with (66) we see that $h_1h_2h_3 = |J| = r^2\sin\phi$.

The Jacobian vanishes when $r = 0$, causing h_2 and h_3 to vanish, but this corresponds to the origin where θ and ϕ are indeterminate. The Jacobian also vanishes when $\phi = 0$ and $\phi = \pi$, corresponding to points on the z-axis where θ is indeterminate. ■

To derive the form of the gradient, divergence, curl, and Laplacian operators in rectangular curvilinear coordinates, it is necessary to introduce the triad of unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ at a general point $(q_1^{(0)}, q_2^{(0)}, q_3^{(0)})$. Here, $\mathbf{e}_1$ is tangent to the q_1 coordinate line, $\mathbf{e}_2$ is tangent to the q_2 coordinate line, and $\mathbf{e}_3$ is tangent to the q_3 coordinate line at the point $(q_1^{(0)}, q_2^{(0)}, q_3^{(0)})$. If we denote a general vector in curvilinear coordinates by $\mathbf{q}(q_1, q_2, q_3)$, the vector forms of the three coordinate lines become

$$\mathbf{q} = \mathbf{q}\big(q_1, q_2^{(0)}, q_3^{(0)}\big), \quad \mathbf{q} = \mathbf{q}\big(q_1^{(0)}, q_2, q_3^{(0)}\big), \quad \text{and} \quad \mathbf{q} = \mathbf{q}\big(q_1^{(0)}, q_2^{(0)}, q_3\big). \tag{67}$$

As a result, the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are, respectively, parallel to the derivatives $\partial\mathbf{q}/\partial q_1$, $\partial\mathbf{q}/\partial q_2$, and $\partial\mathbf{q}/\partial q_3$ at the point $(q_1^{(0)}, q_2^{(0)}, q_3^{(0)})$. The scale factors along these coordinate lines are h_1, h_2, and h_3, it follows that the unit vectors at $(q_1^{(0)}, q_2^{(0)}, q_3^{(0)})$ are

$$\mathbf{e}_1 = \frac{\partial\mathbf{q}}{\partial q_1}\Bigg/\left|\frac{\partial\mathbf{q}}{\partial q_1}\right|, \quad \mathbf{e}_2 = \frac{\partial\mathbf{q}}{\partial q_2}\Bigg/\left|\frac{\partial\mathbf{q}}{\partial q_2}\right|, \quad \text{and} \quad \mathbf{e}_3 = \frac{\partial\mathbf{q}}{\partial q_3}\Bigg/\left|\frac{\partial\mathbf{q}}{\partial q_3}\right|,$$

where, of course, the scale factors h_1, h_2, and h_3 are given by

$$h_1 = \left|\frac{\partial\mathbf{q}}{\partial q_1}\right|, \quad h_2 = \left|\frac{\partial\mathbf{q}}{\partial q_2}\right|, \quad \text{and} \quad h_3 = \left|\frac{\partial\mathbf{q}}{\partial q_3}\right|,$$

so that

$$\mathbf{e}_1 = \frac{1}{h_1}\frac{\partial\mathbf{q}}{\partial q_1}, \quad \mathbf{e}_2 = \frac{1}{h_2}\frac{\partial\mathbf{q}}{\partial q_2}, \quad \mathbf{e}_3 = \frac{1}{h_3}\frac{\partial\mathbf{q}}{\partial q_3}. \tag{68}$$

It is important to recognize that unlike the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, which are parallel to the fixed x-, y-, and z-axes so their derivatives are zero, the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ in curvilinear coordinates are functions of position, so when finding the form of vector operators, we must take into account the derivatives of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$.

THEOREM 11.8

grad, div, and curl in general rectangular curvilinear coordinates

Gradient, divergence, curl, and Laplacian in general rectangular curvilinear coordinates Let the scalar function $f(q_1, q_2, q_3)$, and the vector function

$$\mathbf{F} = F_1(q_1, q_2, q_3)\mathbf{e}_1 + F_2(q_1, q_2, q_3)\mathbf{e}_2 + F_3(q_1, q_2, q_3)\mathbf{e}_3$$

be suitably differentiable functions of the rectangular curvilinear coordinates q_1, q_2, and q_3, where $\mathbf{e}_1$ is the unit vector in the direction of increasing q_1, $\mathbf{e}_2$ is the unit vector in the direction of increasing q_2, and $\mathbf{e}_3$ is the unit vector in the direction of increasing q_3 at the point (q_1, q_2, q_3). Then:

(i) $$\operatorname{grad} f = \mathbf{e}_1\frac{1}{h_1}\frac{\partial f}{\partial q_1} + \mathbf{e}_2\frac{1}{h_2}\frac{\partial f}{\partial q_2} + \mathbf{e}_3\frac{1}{h_3}\frac{\partial f}{\partial q_3}$$

(ii) $$\operatorname{div}\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3F_1) + \frac{\partial}{\partial q_2}(h_1h_3F_2) + \frac{\partial}{\partial q_3}(h_1h_2F_3)\right]$$

(iii) $$\operatorname{curl}\mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}$$

(iv) $$\Delta \equiv \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial q_3}\right)\right]$$

(the Laplacian operator)

Proof

(i) To find grad $f = \frac{\partial f}{\partial x_1}\mathbf{i} + \frac{\partial f}{\partial x_2}\mathbf{j} + \frac{\partial f}{\partial x_3}\mathbf{k}$ in terms of curvilinear coordinates it is necessary to find the components of this vector in the $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ directions, and then to use them as the components of a vector expressed in terms of curvilinear coordinates. As only q_1 varies in the direction of $\mathbf{e}_1$, it follows from the first equations in (46) and (68) that

$$\mathbf{e}_1 = \frac{1}{h_1}\left(\frac{\partial x_1}{\partial q_1}\mathbf{i} + \frac{\partial x_2}{\partial q_1}\mathbf{j} + \frac{\partial x_3}{\partial q_1}\mathbf{k}\right).$$

Thus, the component of grad f in the direction of the unit vector $\mathbf{e}_1$ is

$$\mathbf{e}_1 \cdot \operatorname{grad} f = \frac{1}{h_1}\left(\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial q_1} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial q_1} + \frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial q_1}\right) = \frac{1}{h_1}\frac{\partial f}{\partial q_1},$$

where the last result follows directly from the chain rule.

Corresponding results apply for the components of grad f in the directions of the unit vectors $\mathbf{e}_2$ and $\mathbf{e}_3$, so if we use these results as the components of grad f in curvilinear coordinates, it follows that

$$\operatorname{grad} f = \mathbf{e}_1\frac{1}{h_1}\frac{\partial f}{\partial q_1} + \mathbf{e}_2\frac{1}{h_2}\frac{\partial f}{\partial q_2} + \mathbf{e}_3\frac{1}{h_3}\frac{\partial f}{\partial q_3},$$

and result (i) is established.

In what follows, for conciseness when establishing results (ii) to (iv), the operator notations $\nabla \cdot (\cdot)$ and $\nabla \times (\cdot)$ will be used to signify the divergence and curl operators.

(ii) As $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are orthogonal unit vectors $\mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_3$. By identifying f in (i) with q_1 we see that $\mathbf{e}_1 = h_1\nabla q_1$ and, similarly, by identifying f with q_2 and q_3 it follows that $\mathbf{e}_2 = h_2\nabla q_2$ and $\mathbf{e}_3 = h_3\nabla q_3$, and so $\mathbf{e}_1 = h_2h_3\nabla q_2 \times \nabla q_3$.

To find div $\mathbf{F}$ it is necessary to compute $\nabla \cdot (F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3)$ taking into account the dependence of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ on position. Because of the linearity of the divergence operator, this can be accomplished by taking the divergence of each term in $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$ and then summing the results. The divergence of the first term is given by $\nabla \cdot (F_1\mathbf{e}_1) = \nabla \cdot (F_1h_2h_3\nabla q_2 \times \nabla q_3)$, so using result (iii) of Theorem 11.6, this becomes

$$\nabla \cdot (F_1\mathbf{e}_1) = F_1h_1h_2\nabla \cdot (\nabla q_2 \times \nabla q_3) + (\nabla q_2 \times \nabla q_3) \cdot \nabla(F_1h_1h_2).$$

However, applying result (v) of Theorem 11.7 to the term $\nabla \cdot (\nabla q_2 \times \nabla q_3)$ and using the fact that curl(grad q_2) = curl(grad q_3) = 0 simplifies this result to

$$\nabla \cdot (F_1\mathbf{e}_1) = (\nabla q_2 \times \nabla q_3) \cdot \nabla(F_1h_1h_2),$$

but $\mathbf{e}_1 = h_2h_3\nabla q_2 \times \nabla q_3$, and so

$$\nabla \cdot (F_1\mathbf{e}_1) = \frac{1}{h_2h_3}\mathbf{e}_1 \cdot \nabla(F_1h_2h_3).$$

In the proof of (i) we saw that

$$\mathbf{e}_1 \cdot \operatorname{grad} f = \frac{1}{h_1}\frac{\partial f}{\partial q_1},$$

so identifying f with $F_1h_2h_3$ we find that

$$\nabla \cdot (F_1\mathbf{e}_1) = \frac{1}{h_1h_2h_3}\frac{\partial(F_1h_2h_3)}{\partial q_1}.$$

Corresponding results apply to $\nabla \cdot (F_2\mathbf{e}_2)$ and $\nabla \cdot (F_3\mathbf{e}_3)$, so summing the results we arrive at result (iii).

(iii) To find curl $\mathbf{F}$ it is necessary to compute $\nabla \times (F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3)$, so as curl is a linear operator, we may compute the curl of each term in $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$ and then sum the results. Considering the term $\nabla \times (F_1\mathbf{e}_1)$ and writing $\mathbf{e}_1 = h_1\nabla q_1$, we find that $\nabla \times (F_1\mathbf{e}_1) = \nabla \times (F_1 h_1 \nabla q_1)$. Applying result (iii) of Theorem 11.7 to this last result, we find that

$$\nabla \times (F_1\mathbf{e}_1) = F_1 h_1 \nabla \times (\nabla q_1) - (\nabla q_1) \times (\nabla F_1 h_1),$$

but $\nabla \times (\nabla q_1) = 0$, and so

$$\nabla \times (F_1\mathbf{e}_1) = -(\nabla q_1) \times (\nabla F_1 h_1).$$

Now $\nabla q_1 = \mathbf{e}_1/h_1$, so if we reverse the sign in the preceding result and compensate by interchanging the order of the factors, the result becomes

$$\nabla \times (F_1\mathbf{e}_1) = \left[\mathbf{e}_1\frac{1}{h_1}\frac{\partial(F_1h_1)}{\partial q_1} + \mathbf{e}_2\frac{1}{h_2}\frac{\partial(F_1h_1)}{\partial q_2} + \mathbf{e}_3\frac{1}{h_3}\frac{\partial(F_1h_1)}{\partial q_3}\right] \times \frac{\mathbf{e}_1}{h_1},$$

and so using the orthogonality of the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, which implies $\mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{0}$, $\mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3$, and $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$, this becomes

$$\nabla \times (F_1\mathbf{e}_1) = \mathbf{e}_2\frac{1}{h_1h_3}\frac{\partial}{\partial q_3}(h_1F_1) - \mathbf{e}_3\frac{1}{h_1h_2}\frac{\partial}{\partial q_2}(h_1F_1).$$

Corresponding results exist for $\nabla \times (F_2\mathbf{e}_2)$ and $\nabla \times (F_3\mathbf{e}_3)$, so combining them we find that

$$\begin{aligned}\nabla \times \mathbf{F} = {} & \mathbf{e}_2\frac{1}{h_1h_3}\frac{\partial}{\partial q_3}(h_1F_1) - \mathbf{e}_3\frac{1}{h_1h_2}\frac{\partial}{\partial q_2}(h_1F_1) + \mathbf{e}_3\frac{1}{h_1h_2}\frac{\partial}{\partial q_1}(h_2F_2) \\ & - \mathbf{e}_1\frac{1}{h_2h_3}\frac{\partial}{\partial q_3}(h_2F_2) + \mathbf{e}_1\frac{1}{h_2h_3}\frac{\partial}{\partial q_1}(h_3F_3) - \mathbf{e}_2\frac{1}{h_1h_3}\frac{\partial}{\partial q_2}(h_3F_3).\end{aligned}$$

This last result is seen to be the expansion of the determinant in (iii), so the proof is complete.

(iv) The Laplacian operator

$$\begin{aligned}\Delta &= \nabla \cdot \left[\mathbf{e}_1\frac{1}{h_1}\frac{\partial}{\partial q_1} + \mathbf{e}_2\frac{1}{h_2}\frac{\partial}{\partial q_2} + \mathbf{e}_3\frac{1}{h_3}\frac{\partial}{\partial q_3}\right] \\ &= \operatorname{div}\left[\mathbf{e}_1\frac{1}{h_1}\frac{\partial}{\partial q_1} + \mathbf{e}_2\frac{1}{h_2}\frac{\partial}{\partial q_2} + \mathbf{e}_3\frac{1}{h_3}\frac{\partial}{\partial q_3}\right].\end{aligned}$$

Using result (ii) of the theorem with the operator $\frac{1}{h_1}\frac{\partial}{\partial q_1}$ in place of F_1, the operator $\frac{1}{h_2}\frac{\partial}{\partial q_2}$ in place of F_2 and the operator $\frac{1}{h_3}\frac{\partial}{\partial q_3}$ in place of F_3, we arrive at result (iv).

■

EXAMPLE 11.23

grad, div, curl, and the Laplacian in cylindrical and spherical polar coordinates

Find the forms taken by grad, div, curl, the Laplacian, and the Laplacian operator in (a) cylindrical polar coordinates and (b) spherical polar coordinates.

Solution **(a)** Using the notation of Example 11.22 and the scale factors $h_1 = 1$, $h_2 = r$, and $h_3 = 1$ found in that example, routine calculations show that in

cylindrical polar coordinates, when $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_z\mathbf{e}_z$,

$$\text{grad } f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z$$

$$\text{div}\,\mathbf{F} = \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

$$\text{curl}\,\mathbf{F} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}$$

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \qquad \text{(Laplacian operator).}$$

(b) Again using the notation of Example 11.21 and the scale factors $h_1 = 1$, $h_2 = r\sin\phi$, $h_3 = r$ found in that example, routine calculations show that in **spherical polar coordinates**, when $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi$,

$$\text{grad } f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi$$

$$\text{div}\,\mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(F_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

$$\text{curl}\,\mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\,\mathbf{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta\,F_\phi \end{vmatrix};$$

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

(Laplacian operator). ■

Descriptions of general orthogonal curvilinear coordinates and the form taken by vector operators in different coordinate systems are to be found in references [1.3] and [5.2], whereas applications to continuum mechanics are to be found in reference [5.4] and to hydrodynamics in reference [6.5]. Further information can also be found in Chapters 23 and 24 of reference [G.3].

Summary

After introducing the concept of general orthogonal curvilinear coordinates, this section then derived expressions for grad, div, curl, and the Laplacian operators in terms of these coordinates. Because of the importance of cylindrical and spherical polar coordinates in

applications, these operators were then expressed in terms of cylindrical and spherical polar coordinates.

EXERCISES 11.6

1. Write out the results of Theorem 11.6 using the operator notation $\nabla(.)$, $\nabla \cdot (.)$, $\nabla \times (.)$ in place of grad, div, and curl.
2. Write out the results of Theorem 11.7 using the operator notation $\nabla(.)$, $\nabla \cdot (.)$, $\nabla \times (.)$ in place of grad, div, and curl.
3. Complete the calculations leading to the results of Example 11.22(a) for cylindrical polar coordinates.
4. Complete the calculations leading to the results of Example 11.22(b) for spherical polar coordinates.
5. Show the curvilinear coordinate system defined in the region $q_3 \geq 0$ by the equations $x_1 = q_1 - q_2$, $x_2 = q_1 + q_2$, and $x_3 = \sinh q_3$ is orthogonal. Find the scale factors h_1, h_2, h_3, grad f, and div $\mathbf{F}$.
6. Show that the **parabolic cylindrical coordinates** (u, v, z) defined by the equations $x = \frac{1}{2}(u^2 - v^2)$, $y = uv$, $z = z$ are orthogonal. Find the scale factors h_1, h_2, h_3, and $\nabla^2 f$.
7. Show that the **elliptic cylindrical coordinates** (ξ, η, z) defined by the equations $x = \cosh \xi \cos \eta$, $y = \sinh \xi \sin \eta$, $z = z$ for $0 \leq \xi < \infty$, $-\pi < \eta \leq \pi$, $-\infty < z < \infty$ are orthogonal. Find the scale factors h_1, h_2, h_3 and state the shapes of the surfaces $\xi =$ constant and $\eta =$ constant and find grad f.

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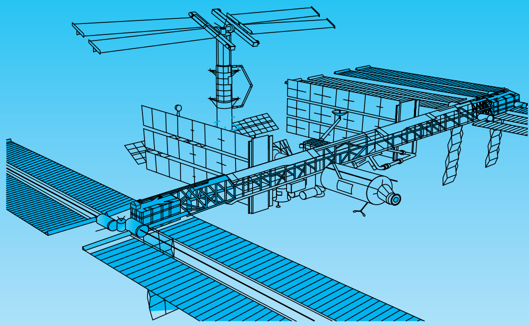
CHAPTER 12



Vector Integral Calculus

When working with the fundamental conservation laws governing engineering and physics, problems often arise that lead to the integral of the divergence of a vector function **F** over a volume V. The Gauss divergence theorem enables the integral of div **F** over volume V to be replaced by the integral of the normal component of **F** over the surface S enclosing V. This result simplifies calculations, because **F** is usually only known in general terms, whereas in physical problems the value of the normal component of **F** on S is known from the conditions of the problem.

Another vector quantity that arises naturally in engineering and physics is the vector function curl **F**, and when this occurs it is often necessary to integrate the normal component of curl **F** over an open surface S. This happens, for example, in fluid mechanics when working with the vorticity and circulation of a fluid. Stokes' theorem replaces the evaluation of the integral of the normal component of curl **F** over the open surface S by a directed line integral of **F** around the curve Γ forming the boundary of S. Here also a simplification results, because once again the vector function **F** on surface S is usually only known in general terms, whereas in physical problems its value on Γ is specified. Green's theorem in the plane is a two-dimensional form of Stokes' theorem, and it has many uses throughout engineering, physics, and mathematics.

The three most important vector integral theorems due to Gauss, Green, and Stokes are derived, followed by the derivation of two important integral transport theorems that play an essential role in mechanics, fluid mechanics, chemical engineering, electromagnetism, and elsewhere. After a review of the background of the vector integral calculus, and an introduction to the concept of an orientable surface, the Gauss divergence theorem and the theorems due to Green and Stokes are proved and applied.

The two fundamental integral transport theorems that are derived and applied are the flux transport theorem, which determines the rate of change of flux passing through an open surface bounded by a moving space curve, and Reynold's transport theorem, which concerns the rate of change of a volume integral when the volume is contained within a moving surface.

12.1 Background to Vector Integral Theorems

Information Provided by Vector Integral Theorems

Physical problems in two and three space dimensions often give rise to integrals with integrands that are determined by a vector field **F** defined over the region of integration. The most important of these integrals involves either the integration of div **F** over a finite volume V, or the integral over a finite open surface S in space of the component of curl **F** normal to S. The objective of this chapter will be to prove some fundamental integral theorems of this type due to Gauss, Stokes, and Green called, respectively, the *Gauss divergence theorem, Stokes' theorem*, and *Green's theorems*. In addition, as optional material, what is called the *flux transport theorem* and the *volume transport theorem* will be proved and, as applications, used to derive some fundamental properties of fluid mechanics.

three important theorems

It will be shown that the **Gauss divergence theorem**, often abbreviated to the **divergence theorem** or **Gauss' theorem**, relates the integral of div **F** over a volume V to the integral over the closed surface S enclosing V of the component of **F** normal to S. Thus, Gauss' theorem allows a volume integral of this type to be replaced by a simpler surface integral. **Stokes' theorem**, which will also be proved in Section 12.2, is of a different nature, in that it relates the integral of the normal component of curl **F** over an open surface S in space bounded by a closed space curve Γ to the line integral of the tangential component of **F** around Γ. So, in the case of Stokes' theorem, a surface integral of a special type over S is related to a simpler line integral around the closed space curve Γ that forms the boundary of S. **Green's theorem in the plane** is the two-dimensional form of Stokes' theorem, and a typical application is to be found in Chapter 14, where it is used in the proof of the Cauchy integral theorem for the integration of complex analytic functions.

Also proved will be two other theorems known as **Green's theorems**, though these results are also known as **Green's identities** or **Green's formulas**. They relate integrals of Laplacians of scalar functions Φ and Ψ over a volume V to the integral over the surface S enclosing V of the derivatives of these functions normal to S. Green's theorems are used extensively when working with partial differential equations involving the Laplacian operator, because they can be used to replace the integral over a volume V of a solution of Laplace's equation that is to be determined by the integral of the normal derivatives of the solution over S that occur as a prescribed boundary condition that must be satisfied by the solution.

A common feature of these theorems is that each frequently replaces an integral of a special type over a region (a volume or an open surface) by a simpler integral over the boundary of the region (a closed surface or a closed space curve), thereby reducing by one the number of dimensions involved in the integration. The integral can then be evaluated by using whichever of the two equivalent expressions is easier. When used with partial differential equations involving the Laplacian operator, Green's theorems typically allow integrals of unknown functions over a region to be replaced by simpler integrals of known functions over the boundary of the region.

The two transport theorems proved in Section 12.3 relate to the determination of the derivative with respect to time of surface and volume integrals of time-dependent integrands when the surface or volume involved moves with time. The *flux* of a vector **F** across a surface S is the integral over S of the component of

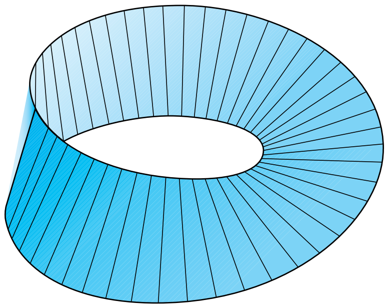


FIGURE 12.1 A Möbius strip.

F normal to S. The flux transport theorem describes the rate of change of the flux of **F** across S, taking into account the time dependence of **F** and the motion of S. A typical example of this type occurs when current is induced in a coil of wire moving in a magnetic field, because the current depends on the rate of change of magnetic flux through the moving coil.

The volume transport theorem describes the time rate of change of a volume integral due to the time dependence of the integrand and the motion of the volume over which integration takes place. A typical application of this theorem arises in fluid mechanics where the boundary of a volume of interest relating to a certain feature of the fluid flow does not move in the same way as the fluid, so that a flow takes place through the surface that encloses the volume.

Surfaces and Orientation

Section 12.2 is concerned with surfaces that have *two* sides and makes use of the normal at each point on such surfaces. It might seem unnecessary to define two-sided surfaces, but it is necessary because pathological surfaces exist that only have *one* side, and these must be excluded from the theorems of Section 12.2.

An example of a one-sided surface is provided by the **Möbius strip** shown in Fig. 12.1. This strip can be considered to be formed from a long strip of paper, the ends of which are joined after making a 180° twist in the paper about its longitudinal center line. Its one-sided nature can easily be verified by drawing a pencil line around the center line of the strip, because eventually the line will connect with the starting point, and if the strip is cut and opened out, examination will show a pencil line on both sides of the paper.

When deriving the Gauss divergence theorem, it will be necessary to work with a closed two-sided surface S, the *interior* of which contains the volume V of space that will concern us. A vector element of area of such a surface will have magnitude dS and an associated unit vector $\mathbf{n}$ normal to dS. As the normal $\mathbf{n}$ at a point on a two-sided surface S enclosing a volume V may be directed away from either side of S, it is necessary to adopt a standard convention for the direction of $\mathbf{n}$ and the vector element of area $d\mathbf{S} = \mathbf{n}dS$ on S. The normal $\mathbf{n}$ at a point on such a surface will always be chosen to be directed *out* of V. So if, for example, V is a sphere, the normal $\mathbf{n}$ at any point of its surface will be along a radial line drawn *outward* from the center of the sphere.

open surfaces and orientable surfaces

A two-sided **open surface** S bounded by a non-self-intersecting space curve Γ is a surface that does *not* have an interior, and so does *not* enclose a volume V. When